ON THE MARX CONJECTURE FOR A CLASS OF CLOSE-TO-CONVEX FUNCTIONS

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We let $G(z) \prec H(z)$ $(|z| < R)$ mean that $G(z)$ is subordinate to $H(z)$ in $|z| < R$ in the sense that $G(z)$ and $H(z)$ are regular in the disk $|z| < R$, and for each fixed $r < R$, the image of the disk $|z| \leq r$ under $G(z)$ is contained in its image under $H(z)$. Let S_α^* $(0 \leq \alpha < 1)$ denote the class of functions that are starlike of order α in the open unit disk $E = \{z: |z| < 1\}$; that is, $f(z)$ belongs to S_α^* if and only if $f(z)$ is regular in E, $f(0) = 0$, $f'(0) = 1$, and

$$\Re \{zf'(z)/f(z)\} \geq \alpha$$

for every z in E.

In 1932, A. Marx [3] conjectured that for every $f(z) \in S_0^*$, the function $f'(z)$ is subordinate to $k'(z)$ in E, where $k(z) = z/(1 - z)^2$ is the Koebe function. B. Pinchuk [5] and R. McLaughlin [4] have studied the corresponding conjecture for the classes S_α^*, namely that $f'(z) \prec k_\alpha'(z)$ in E for every $f(z) \in S_\alpha^*$, where

$$k_\alpha(z) = z/(1 - z)^{2-2\alpha}.$$

For each $\alpha \in [0, 1)$, McLaughlin [4] has determined a number r_α $(0 < r_\alpha < 1)$ such that the Marx conjecture for S_α^* holds in the disk $|z| \leq r_\alpha$. The constant $r_0 = 0.736 \cdots$ in [4] had been discovered earlier by P. L. Duren [1]. For $\alpha = 1/2$, it was shown that $r_{1/2} = 0.81046 \cdots$ [4]. In this note, we consider a class of close-to-convex functions that contains $S_{1/2}^*$ as a proper subclass, and we show that for every $f(z)$ in this class, the relation $f'(z) \prec k_{1/2}'(z)$ holds in the entire disk E.

For $0 \leq \alpha < 1$ and $0 \leq \beta < 1$, we say that $f(z) \in \mathcal{K}(\alpha, \beta)$ if and only if

(i) $f(z)$ is regular in E, $f(0) = 0$, $f'(0) = 1$, and

(ii) for some $g(z) \in S_\beta^*$,

$$\Re \{zf'(z)/g(z)\} \geq \alpha \tag{1}$$

for every z in E. We note that $\mathcal{K}(\alpha, \beta)$ is a subclass of the class of close-to-convex functions of order α and type β introduced by R. J. Libera [2]. Instead of condition (ii), Libera required that (1) hold for some $g(z)$ such that $ag(z) \in S_\beta^*$ for some complex number a of modulus 1 [2, Definition (1.2)]. The class S_α^* is the subset of functions $f(z) \in \mathcal{K}(\alpha, \alpha)$ such that $g(z) = f(z)$ in (1).

We shall need the Herglotz representations for the classes S_β^* and $\mathcal{K}(\alpha, \beta)$. Let I denote the class of nondecreasing functions with total variation 1 on the interval $[0, 2\pi]$. It is well known that $g \in S_\beta^*$ if and only if

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$$g(z) = z \exp\left\{ -2(1-\beta) \int_0^{2\pi} \log(1 - ze^{-it})\, dp(t) \right\} \tag{2}$$

for some $p(t) \in I$ (see, for example, [5]). If $f \in \mathscr{K}(\alpha, \beta)$, then

$$zf'(z)/g(z) = \alpha + (1-\alpha) P(z), \tag{3}$$

where $g \in S_\beta^*$ and $P(z) = 1 + c_1 z + \cdots$ is regular and has positive real part in E. By the Herglotz theorem, there exists a function $m(t) \in I$ such that

$$P(z) = \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}}\, dm(t) = -1 + \int_0^{2\pi} \frac{2}{1 - ze^{-it}}\, dm(t). \tag{4}$$

We shall simplify the appearance of our formulas by using the notation

$$\ell(z, t) = \frac{1}{1 - ze^{-it}} \quad \text{and} \quad L(z, t) = \log \ell(z, t) = \log \frac{1}{1 - ze^{-it}}.$$

By (2), (3), and (4), we have that $f \in \mathscr{K}(\alpha, \beta)$ if and only if there exist functions $m(t)$ and $p(t)$ in I such that

$$\begin{aligned} f'(z) &= \left\{ 2\alpha - 1 + 2(1-\alpha) \int_0^{2\pi} \ell(z, t)\, dm(t) \right\} \exp\left\{ 2(1-\beta) \int_0^{2\pi} L(z, t)\, dp(t) \right\} \\ &= (2\alpha - 1) \frac{g(z)}{z} + 2(1-\alpha) \frac{g(z)}{z} \int_0^{2\pi} \ell(z, t)\, dm(t). \end{aligned} \tag{5}$$

THEOREM. *Let* $f \in \mathscr{K}(\alpha, \beta)$, *and let* g *be a function in* S_β^* *such that*

$$\Re \{zf'(z)/g(z)\} \geq \alpha \qquad (z \in E).$$

Then

$$\frac{f'(z) + (1 - 2\alpha) g(z)/z}{2(1-\alpha)} \prec \frac{1}{(1-z)^{3-2\beta}} \quad \textit{in } E. \tag{6}$$

Proof. We rearrange terms in (5) and take the logarithm to obtain

$$\begin{aligned} &\log\left[\frac{f'(z) + (1-2\alpha) g(z)/z}{2(1-\alpha)} \right] \\ &= (3 - 2\beta)\left[\frac{1}{3-2\beta} \log\left\{ \int_0^{2\pi} \ell(z, t)\, dm(t) \right\} + \frac{2(1-\beta)}{3-2\beta} \int_0^{2\pi} L(z, t)\, dp(t) \right]. \end{aligned} \tag{7}$$

For each z in the disk $|z| \leq r$ and any $m(t) \in I$, the integral $\int_0^{2\pi} \ell(z, t)\, dm(t)$ is a complex number in the closed disk bounded by the circle

$$C_r = \{\ell(r, t) \colon 0 \leq t \leq 2\pi\}.$$

Hence $\log\left\{\int_0^{2\pi} \ell(z, t)\,dm(t)\right\}$ is in Ω_r, the closed convex set bounded by the curve $\gamma_r = \{L(r, t)\colon 0 \leq t \leq 2\pi\}$ (γ_r is convex, since $\log(1 - z)$ is a convex function on E). Similarly, the convexity of γ_r implies that

$$\int_0^{2\pi} L(z, t)\,dp(t) \in \Omega_r$$

for every z in $|z| \leq r$ and every $p(t) \in I$. Therefore, for all z in $|z| \leq r$ and $p(t), m(t) \in I$, the quantity in brackets on the right in (7) is a point in Ω_r, since it is a convex combination of two points in the convex set Ω_r. Hence, by (7), we have that

$$\log\left[\frac{f'(z) + (1 - 2\alpha)g(z)/z}{2(1 - \alpha)}\right] \prec (3 - 2\beta)\log\frac{1}{1 - z} \tag{8}$$

in E. Exponentiation of both sides in (8) preserves the subordination relation and yields the desired conclusion (6).

COROLLARY 1. *If* $f \in \mathscr{K}(1/2, 1/2)$, *then*

$$f'(z) \prec (1 - z)^{-2} = k'_{1/2}(z) \quad \textit{in } E.$$

Since $S^*_{1/2}$ is a proper subset of $\mathscr{K}(1/2, 1/2)$, the result of Corollary 1 includes the Marx conjecture for $S^*_{1/2}$. The validity of the Marx conjecture for $S^*_{1/2}$ also follows from the inequality

$$\Re\sqrt{f'(z)} > 1/2 \qquad (f \in S^*_{1/2},\ z \in E)$$

proved by Marx [3, p. 62].

COROLLARY 2. *If* $f \in S^*_\alpha$ *and* $\lambda = 1/(2 - 2\alpha)$, *then*

$$\lambda f'(z) + (1 - \lambda)\frac{f(z)}{z} \prec \frac{1}{(1 - z)^{3-2\alpha}} = \lambda k'_\alpha(z) + (1 - \lambda)\frac{k_\alpha(z)}{z} \tag{9}$$

in E.

Remarks. Since the representation formula (2) implies that $f(z)/z$ is subordinate to $k_\alpha(z)/z$ ($f \in S^*_\alpha$), it would seem that (9) should be helpful in settling the Marx conjecture for S^*_α; but we have made no progress with this line of reasoning for $\alpha \neq 1/2$. Although (9) may not lead to a proof of the Marx conjecture, it may be possible to improve the results of McLaughlin [4] for α near 1/2 (λ near 1) with the aid of (9). We also note that the two sides of (9) are convex combinations ($0 \leq \lambda \leq 1$) of the functions involved only for $0 \leq \alpha \leq 1/2$. For starlike functions, (9) yields the interesting relation

$$f'(z) + f(z)/z \prec k'(z) + k(z)/z \qquad (f \in S^*_0,\ z \in E).$$

Finally, we mention that replacement of $\mathscr{K}(\alpha, \beta)$ in our theorem by Libera's more general class of close-to-convex functions of order α and type β yields a theorem, where (6) is replaced by the conclusion

$$\frac{f'(z) + (e^{i\phi} - 2\alpha)g(z)/z}{2(\cos\phi - \alpha)e^{i\phi}} \prec \frac{1}{(1 - z)^{3-2\beta}} \quad \text{in } E.$$

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