SETS OF UNIQUENESS ON THE PRODUCT OF COMPACT GROUPS

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Let $X = \prod_{n=1}^{\infty} X_n$ be the product of countably many non-Abelian, compact, topological groups, and let μ denote Haar measure on X. Let f_n be a coordinate function of $V^{(n)}$, where $V^{(n)}$, different from 1, is a continuous, unitary, irreducible representation (CUIR) of X_n. For $\bar{x} = (x_1, x_2, \cdots)$ in X, define $f_n(\bar{x})$ to be $f_n(x_n)$.

A subset C of X is called a *set of uniqueness* with respect to a regular method S of summability (or briefly, a U_S-set) if S-summability to 0 on the complement of C of a series $\sum c_n f_n$ with complex coefficients c_n implies that $c_n = 0$ for each n. Otherwise, C is called a *set of multiplicity* (an M_S-set).

Let d_n be the dimension of the representation space of $V^{(n)}$, and set $M = \sup \{d_n : n \geq 1\}$. We prove that if $M < \infty$ and $\mu(C) < 1/2M$, then C is a U_S-set. If $M = \infty$, then every subset of X of measure 0 is a set of uniqueness. We also demonstrate that if each X_n is connected, then every subset of X of measure less than 1/2 is a U_S-set.

1. PRELIMINARIES

Let μ_n and μ denote normalized Haar measure on X_n and X, respectively, and write the identity element of X as $\bar{e} = (e_1, e_2, \cdots)$, where e_n is the identity in X_n.

For each n, choose α_n in X_n so that there exists a continuous, unitary, irreducible representation $V^{(n)}$ of X_n, on a Hilbert space H_n of dimension $d_n \geq 2$, for which $V^{(n)}_{\alpha_n} \neq I$. (That such a choice is possible in every compact non-Abelian group G can be demonstrated as follows. Let $a, b \in G$ be such that $ab \neq ba$; then $aba^{-1}b^{-1}$ is not the identity in G. By [2, (22.12)], we can find a CUIR V of G such that $V_{aba^{-1}b^{-1}} \neq I$. If V were one-dimensional, we could conclude that $V_{aba^{-1}b^{-1}} = V_a V_b V_a^{-1} V_b^{-1} = I$, contrary to our choice of V.) It follows that $\alpha_n \neq e_n$.

For $V^{(n)}$, let $\{\xi_1^{(n)}, \cdots, \xi_{d_n}^{(n)}\}$ be an orthonormal basis of H_n such that

$$V^{(n)}_{\alpha_n} \xi_k^{(n)} = \lambda_k^{(n)} \xi_k^{(n)} \qquad \text{for } k = 1, 2, \cdots, d_n,$$

where $|\lambda_k^{(n)}| = 1$. Since $V^{(n)} \neq I$, there exists an element $q \in \{1, 2, \cdots, d_n\}$ for which $\lambda_q^{(n)} \neq 1$. For such a q and arbitrary $p \neq q$, define the complex-valued

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function f_n on X by $f_n(\bar{x}) = v_{pq}^{(n)}(x_n)$, where $v_{pq}^{(n)}(x_n)$ is the coordinate function $\left\langle V_{x_n}^{(n)} \xi_q^{(n)}, \xi_p^{(n)} \right\rangle$. The resulting set $\{f_n\}$ is orthogonal.

Definition. Let $S = (a_{ij})$ $(i, j = 1, 2, \cdots)$ denote a regular method of summability, and let the sequence $\{f_n\}$ have the properties listed in the introduction. Corresponding to each sequence $\{c_n\}$ of complex numbers, we define the set

$$E^{(S)}(c_n) = \left\{ \bar{x} \in X\colon \sum_{n=1}^{\infty} c_n f_n(\bar{x}) \text{ is S-summable to } 0 \right\}.$$

For each element $\bar{y}$ in X, define

$$E^{(S)}(c_n) \cdot \bar{y} = \{\bar{x} \cdot \bar{y}\colon \bar{x} \in E^{(S)}(c_n)\}.$$

When confusion seems unlikely, we shall denote these sets by $E^{(S)}$ and $E^{(S)} \cdot \bar{y}$.

2. SETS OF UNIQUENESS IN X

We now prove a principal theorem from which we derive the existence of uniqueness sets in X. Our restriction on the measure of $E^{(S)}(c_n)$ is necessary: without it, the theorem is false, as we demonstrate in the discussion that follows Corollary 2. It is convenient to state the following lemma (we omit its proof).

LEMMA 1. *If* $S = (a_{ij})$ *is a regular method of summability, then*

$$\lim_{m \to \infty} \sum_{i=p}^{\infty} a_{mi} = 1$$

for $p = 1, 2, \cdots$.

THEOREM 1. *Suppose that* $M = \sup\{d_n\colon n \geq 1\} < \infty$. (By the previous discussion, $M \geq 2$.) *Let* $\{c_n\}_{n=1}^{\infty}$ *be a sequence of complex numbers, and suppose that* $E^{(S)}(c_n)$ *has Haar measure exceeding* $1 - 1/2M$. *Then* $c_n = 0$ *for* $n = 1, 2, \cdots$.

Proof. For the fixed elements α_n in X_n defined in the previous section, set $\bar{\alpha}_n = (e_1, e_2, \cdots, e_{n-1}, \alpha_n, e_{n+1}, \cdots)$. Since $1 - 1/2M \geq 3/4$, we infer that $E^{(S)} \cap E^{(S)} \cdot \bar{\alpha}_n \neq \emptyset$ for each $n \geq 1$. In fact, writing

$$\mu(E^{(S)} \cap E^{(S)} \cdot \bar{\alpha}_n) = \mu(E^{(S)}) - \mu(E^{(S)} \cap (X \setminus E^{(S)} \cdot \bar{\alpha}_n))$$

and noting that $\mu(X \setminus E^{(S)} \cdot \bar{\alpha}_n) = \mu(X \setminus E^{(S)}) < 1/2M$, we conclude that $\mu(E^{(S)} \cap E^{(S)} \cdot \bar{\alpha}_n) > 1 - 1/M$.

Let r be a positive integer. We claim that there exists some $\bar{z} \in E^{(S)} \cap E^{(S)} \cdot \bar{\alpha}_r$ $(\bar{z} = \bar{y} \cdot \bar{\alpha}_r)$ with $\bar{y}$ in $E^{(S)}$, for which $f_r(\bar{y} \cdot \bar{\alpha}_r) \neq 0$, that is, $v_{pq}^{(r)}(y_r \cdot \alpha_r) \neq 0$. For suppose not; then $v_{pq}^{(r)}(y_r \cdot \alpha_r) = 0$ for every $\bar{y} \in E^{(S)}$ such that $\bar{y} \cdot \bar{\alpha}_r \in E^{(S)}$. Write F_r for $E^{(S)} \cap E^{(S)} \cdot \bar{\alpha}_r$. Let $P_k = P_k(F_r)$ be the projection into X_k of F_r; that is, let

$$P_r = \{y_r \cdot \alpha_r \in X_r\colon \text{there exists } \bar{y} \in E^{(S)} \text{ with } r^{\text{th}} \text{ coordinate } y_r \text{ such that } \bar{y} \cdot \bar{\alpha}_r \in E^{(S)}\}.$$

Under our assumption, $v_{pq}^{(r)}$ is zero on P_r. By the first paragraph, $\mu(F_r) > 1 - 1/M$; since $F_r \subset \prod_{k=1}^{\infty} P_k$, we conclude that $\mu(P_r) > 1 - 1/M$. Consequently,

$$1/d_r = \int_{X_r} |v_{pq}^{(r)}|^2 d\mu_r = \int_{X_r \setminus P_r} |v_{pq}^{(r)}|^2 d\mu_r \leq \mu_r(X_r \setminus P_r) < 1/M .$$

Therefore $d_r > M$, a contradiction.

Thus there exists a $\bar{y} \in E^{(S)}$ such that $\bar{y} \cdot \bar{\alpha}_r \in E^{(S)}$ and $v_{pq}^{(r)}(y_r \cdot \alpha_r) \neq 0$. Since $v_{pq}^{(r)}(y_r \cdot \alpha_r) = \lambda_q^{(r)} \cdot v_{pq}^{(r)}(y_r)$ and since $\lambda_q^{(r)} \neq 1$, we infer that $f_r(\bar{y} \cdot \bar{\alpha}_r) \neq f_r(\bar{y})$.

By definition of $E^{(S)}$, $\sigma_m(\bar{y}) \to 0$ and $\sigma_m(\bar{y} \cdot \bar{\alpha}_r) \to 0$ as $m \to \infty$, where, if S is given by the matrix (a_{ij}),

$$\sigma_m = \sum_{i=1}^{\infty} a_{mi} s_i \quad \text{and} \quad s_i = \sum_{n=1}^{i} c_n f_n .$$

Assume not all the coefficients c_n are 0, and let q be the least integer for which $c_q \neq 0$. By the previous paragraph, there is an element $\bar{z} \in E^{(S)} \cap E^{(S)} \cdot \bar{\alpha}_q$ $(\bar{z} = \bar{y} \cdot \bar{\alpha}_q)$ with $\bar{y} \in E^{(S)}$ for which $f_q(\bar{y} \cdot \bar{\alpha}_q) \neq f_q(\bar{y})$. It satisfies the relations

$$\begin{aligned}\sigma_m(\bar{y} \cdot \bar{\alpha}_q) &= \sum_{i=1}^{\infty} a_{mi} s_i(\bar{y} \cdot \bar{\alpha}_q) = \sum_{i=q}^{\infty} a_{mi} \left[\sum_{n=q}^{i} c_n f_n(\bar{y} \cdot \bar{\alpha}_q) \right] \\ &= \sum_{i=q}^{\infty} a_{mi} \left[c_q f_q(\bar{y} \cdot \bar{\alpha}_q) - c_q f_q(\bar{y}) + \sum_{n=q}^{i} c_n f_n(\bar{y}) \right] \\ &= c_q [f_q(\bar{y} \cdot \bar{\alpha}_q) - f_q(\bar{y})] \cdot \sum_{i=q}^{\infty} a_{mi} + \sigma_m(\bar{y}) .\end{aligned}$$

Passing to the limit as $m \to \infty$ and using Lemma 1, we see that

$$c_q [f_q(\bar{y} \cdot \bar{\alpha}_q) - f_q(\bar{y})] = 0 .$$

Since $f_q(\bar{y} \cdot \bar{\alpha}_q) \neq f_q(\bar{y})$, we conclude that $c_q = 0$, a contradiction. Thus each coefficient is zero, and the theorem is proved.

COROLLARY 1. *Suppose that* $M = \sup \{d_n : n \geq 1\} < \infty$, *and let* C *be a subset of* X *of measure less than* $1/2M$. *Then* C *is a* U_S*-set for each regular summability method* S.

THEOREM 2. *Let* M *be as above, and suppose that* $M = \infty$. *If* $E^{(S)}(c_n)$ *has Haar measure* 1, *then every* c_n *is zero.*

Proof. We retain the notation of Theorem 1. For every (fixed) positive integer n, we have the relation $\mu(E^{(S)}) = \mu(E^{(S)} \cdot \bar{\alpha}_n) = 1$. It follows that

$$\mu(E^{(S)} \cap E^{(S)} \cdot \bar{\alpha}_n) = 1 ,$$

whence $\mu_k(P_k) = 1$ for all k; in particular, $\mu_n(P_n) = 1$.

For each $n \geq 1$, we shall show, as in the proof of Theorem 1, that there exists $\bar{z} \in E^{(S)} \cap E^{(S)} \cdot \bar{\alpha}_n$ ($\bar{z} = \bar{y} \cdot \bar{\alpha}_n$) with $\bar{y} \in E^{(S)}$, for which $f_n(\bar{y} \cdot \bar{\alpha}_n) \neq 0$; that is, $v_{pq}^{(n)}(y_n \cdot \alpha_n) \neq 0$. If this is not the case, then $v_{pq}^{(n)}$ is zero on P_n, and hence

$$1/d_n = \int_{X_n} |v_{pq}^{(n)}|^2 \, d\mu_n = \int_{X_n \setminus P_n} |v_{pq}^{(n)}|^2 \, d\mu_n \leq \mu_n(X_n \setminus P_n) = 0,$$

a contradiction. Thus such an element $\bar{z}$ exists. The proof is now concluded exactly as in Theorem 1.

COROLLARY 2. *Let* M *be defined as in Corollary* 1, *and suppose that* $M = \infty$. *Then every subset of* X *of measure* 0 *is a* U_S*-set.*

The essence of the proof of Theorem 1 is that the coordinate function $v_{pq}^{(n)}$ cannot be zero on a subset of the group X_n, if that set has large enough measure. Two questions arise in this connection:

(i) Let G be a compact topological group, with normalized Haar measure μ, and suppose that V is a CUIR of G on a Hilbert space H, with orthonormal basis $\{\xi_1, \cdots, \xi_d\}$. Can it happen that for each pair of indices p and q in $\{1, 2, \cdots, d\}$, the group G contains a set C of positive measure such that $v_{pq}(x) = \langle V_x \xi_q, \xi_p \rangle = 0$ for each x in C? Can each of the sets C have measure greater than 1/2?

(ii) More generally, can $v_{pq}(x)$ be constant on such a set C?

The proof of Theorem 1 indicates that the answer to (i) is negative if the measure of C is large enough. Further, if H has dimension $d = 2$, then v_{pq} cannot assume the value zero on a set C of measure greater than 1/2, for this would imply that

$$1/2 = 1/d = \int_G |v_{pq}|^2 \, d\mu = \int_{G \setminus C} |v_{pq}|^2 \, d\mu \leq \mu(G \setminus C) < 1/2 .$$

If $d \geq 3$, however, the answer to (i) is affirmative. The alternating group A_4 admits such a representation V, of dimension 3, for which the coordinate function v_{33} vanishes on a set of measure 2/3 (see [3, p. 49]).

In the case $d = 2$, it is possible for v_{pq} to be constant (necessarily nonzero) on a set of measure greater than 1/2: there exists a CUIR V of the symmetric group S_3 for which v_{22} assumes the value $-1/2$ on a set of measure 2/3.

3. SETS OF UNIQUENESS ON CONNECTED PRODUCT SPACES

Next we investigate the conditions under which the conclusion of Theorem 1, and hence that of Corollary 1, can be made independent of the constant M, that is, independent of the dimension of the representation space of each $V^{(n)}$. To this end, we shall find it expedient to determine when the set $\{x \in G: \langle V_x \xi_q, \xi_p \rangle = 0\}$ has Haar measure 0, where V is a CUIR of the compact group G. Theorem 3 shows that it is sufficient to assume that G is connected. For the proof of this result, see [1].

THEOREM 3. *Let* G *be a non-Abelian compact topolotical group, with (normalized) Haar measure* μ, *and let* V *be a continuous, unitary, irreducible representation of* G, *with* $V \neq 1$. *Define the set* $C = \{x \in G: \langle V_x \xi_q, \xi_p \rangle = 0\}$, *where* ξ_p *and* ξ_q *denote elements in an orthonormal basis for the representation space of* V.

If G *is connected, then* $\mu(C) = 0$ *for every choice of* ξ_p *and* ξ_q.

With the aid of Theorem 3, we can prove that the conclusions of Theorem 1 and Corollary 1 remain valid with M replaced by 1 if the product space X is connected. We first prove the following lemma.

LEMMA 2. *Let* X_n *and* X *satisfy the conditions in the introduction, and suppose that each* X_n *is connected. If* $E^{(S)}(c_n)$ *has Haar measure greater than* 1/2, *then each* c_n *is zero.*

Proof. Define $\bar{\alpha}_n$ as in the proof of Theorem 1. Since $\mu(E^{(S)}) > 1/2$, the inequality $\mu(E^{(S)} \cap E^{(S)} \cdot \bar{\alpha}_n) > 0$ holds for each n, and it follows from Theorem 3 that f_n cannot vanish throughout $E^{(S)} \cap E^{(S)} \cdot \bar{\alpha}_n$. Thus there exists an element $\bar{y} \cdot \bar{\alpha}_n$ in $E^{(S)}$, with $\bar{y}$ in $E^{(S)}$, such that $f_n(\bar{y} \cdot \bar{\alpha}_n) \neq 0$. It follows that $f_n(\bar{y} \cdot \bar{\alpha}_n) \neq f_n(\bar{y})$, and this implies, as in the proof of Theorem 2, that c_n is zero.

COROLLARY 3. *With the notation as in Lemma* 2, *suppose that each* X_n *is connected, and let* C *be a subset of* X *of measure less than* 1/2. *Then* C *is a set of uniqueness for each regular method of summability.*

Because a series that converges to a value B is S-summable to B, for each regular method S of summability (see, for example, [4, vol. I, p. 74]), it follows from the definition that a U_S-set is a set of uniqueness with respect to ordinary convergence (called a U-set). Similarly, an M-set is an M_S-set for each regular method S.

With the aid of the following theorem, we shall show, in Theorem 5, that the notions of U-set and U_S-set coincide if the product space is connected. (Although we shall not prove it here, we mention in passing that the converse is also true; that is, if X is not connected, then there exists a U-set that is not a U_S-set for each method S.)

THEOREM 4. *Let* X *be the product of countably many compact and connected non-Abelian groups, and suppose that the set* $E^{(S)}(c_n)$ *has positive Haar measure in* X. *Then there exists a positive integer* N *such that* c_n *is zero for every* $n \geq N$.

Proof. The function $\bar{x} \to \mu(E^{(S)} \cap E^{(S)} \cdot \bar{x})$ is continuous on X (see [2, (20.17)]). Since $\mu(E^{(S)})$ is positive, it follows that $\mu(E^{(S)} \cap E^{(S)} \cdot \bar{x})$ is positive for all $\bar{x}$ in a sufficiently small neighborhood of the identity $\bar{e}$ in X. By the way in which the elements $\bar{\alpha}_r$ were defined, we conclude that $\mu(E^{(S)} \cap E^{(S)} \cdot \bar{\alpha}_r)$ is positive for all sufficiently large values of r, say $r \geq N$.

As before, let P_k denote projection into X_k, and write E_r for $E^{(S)} \cap E^{(S)} \cdot \bar{\alpha}_r$. Then $\mu_k(P_k(E_r)) > 0$ for each $r \geq N$ and every k; in particular, $\mu_r(P_r(E_r)) > 0$. Since $P_r(E_r)$ has positive measure, $v_{pq}^{(r)}$ cannot vanish throughout $P_r(E_r)$, in view of Theorem 3. Thus E_r contains an element $\bar{z} = \bar{y} \cdot \bar{\alpha}_r$, with $\bar{y} \in E^{(S)}$, such that $f_r(\bar{z}) \neq 0$, whence $f_r(\bar{y} \cdot \bar{\alpha}_r) \neq f_r(\bar{y})$.

Since $\bar{y}$ and $\bar{y} \cdot \bar{\alpha}_r$ differ only in the r^{th} coordinate, we have the relations

$$\sum_{n=1}^{i} c_n [f_n(\bar{y} \cdot \bar{\alpha}_r) - f_n(\bar{y})] = 0 \qquad (i < r),$$

$$\sum_{n=1}^{i} c_n [f_n(\bar{y} \cdot \bar{\alpha}_r) - f_n(\bar{y})] = c_r [f_r(\bar{y} \cdot \bar{\alpha}_r) - f_r(\bar{y})] \qquad (i \geq r).$$

Thus, for all m, we obtain the formula

$$\sigma_m(\bar{y}\cdot\bar{\alpha}_r) - \sigma_m(\bar{y}) = \sum_{i=1}^{\infty} a_{mi}\left[\sum_{n=1}^{i} c_n\,(f_n(\bar{y}\cdot\bar{\alpha}_r) - f_n(\bar{y}))\right]$$

$$= \left(\sum_{i=r}^{\infty} a_{mi}\right)\cdot c_r\,[f_r(\bar{y}\cdot\bar{\alpha}_r) - f_r(\bar{y})].$$

Passing to the limit as $m \to \infty$ and appealing to Lemma 1, we conclude that $c_r = 0$. Because $r \geq N$ is arbitrary, the proof is complete.

THEOREM 5. *Let* X *be the product of countably many compact and connected non-Abelian groups, and suppose that* C *is a subset of* X *of measure less than* 1. *The following statements are equivalent:*

(i) C *is a* U-*set* (M-*set*);

(ii) C *is a* U_S-*set* (M_S-*set*) *for some regular summability method* S;

(iii) C *is a* U_S-*set* (M_S-*set*) *for every regular method* S.

Proof. That (i) is equivalent to (ii) for both U-sets and M-sets follows from the facts that S-summability to a value B implies convergence to B and that ordinary convergence is a regular summability method. To prove the theorem for sets of uniqueness, we must show that (ii) implies (iii). Thus let S and T be regular methods of summability, and suppose that C is a U_S-set. If a series is T-summable to 0 on $X \setminus C$, it follows from Theorem 4 that this series is finite. Thus the series converges to 0 on $X \setminus C$ and is therefore S-summable to 0 on this set. Therefore each coefficient is zero, and hence C is a U_S-set.

To show that (ii) implies (iii) for sets of multiplicity, we simply note that since the complement of C has positive measure, every series that is S-summable to 0 outside C must be finite and is therefore summable to the same value by all regular methods.

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