NONRETRACTABLE CUBES-WITH-HOLES

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1. INTRODUCTION AND DEFINITIONS

A *cube-with-handles of genus* n is a compact, orientable 3-manifold that is the regular neighborhood of a finite, connected graph having Euler characteristic $1 - n$. If M is a cube-with-handles of genus n embedded as a polyhedral subset of the 3-sphere S^3, then S^3 - Int M is called a *cube-with-holes of genus* n. A cube-with-holes N is said to be *retractable* if N can be retracted onto a wedge of n simple closed curves, where n is the genus of N. Otherwise, we say the cube-with-holes N is *nonretractable*. If N is a retractable cube-with-holes of genus n and N can be retracted onto a wedge of n simple closed curves in Bd N, then we say N is boundary-retractable.

In [4], Jaco and D. R. McMillan gave examples of cubes-with-holes of genus n, for every $n \geq 2$, that are retractable but not boundary-retractable. Their examples are the same as the examples that Lambert [5] used to show that for every $n \geq 2$ there exists a cube-with-holes N_n, of genus n, such that no mapping of N_n onto a cube-with-handles H_n, of genus n, takes Bd N_n homeomorphically onto Bd H_n. The existence of such a mapping from N_n to H_n is equivalent to the boundary-retractability of N_n [4, p. 153, Theorem 3]. Jaco and McMillan also gave examples of nonretractable cubes-with-holes of genus n, for every $n \geq 3$. However, they were unable to resolve the question in the case of genus 2.

In Section 2 we show that there exists a nonretractable cube-with-holes of genus 2. Using this example, we are able to construct nonretractable cubes-with-holes of genus n, for each $n \geq 2$.

If G is a group and $a, b \in G$, we denote the *commutator* $a^{-1}b^{-1}ab$ *of* a *and* b by $[a, b]$. For subsets A and B of the group G, we use $[A, B]$ to denote the subgroup of G generated by the set of all commutators $[a, b]$ with $a \in A$ and $b \in B$. Let $G_1 = G$, and define $G_{m+1} = [G_m, G]$ for each $m \geq 1$. The group G_2 is called the *commutator subgroup* of G. The series $G_1 \supseteq G_2 \supseteq \cdots \supseteq G_m \supseteq G_{m+1} \supseteq \cdots$ is called the *lower central series* of the group G. We use the notation

$$G_\omega = \bigcap_{m \geq 1} G_m .$$

In Section 3 we show that if N *is a retractable cube-with-holes with fundamental group* G, *then* N *is boundary-retractable if and only if the natural homomorphism*

$$\pi_1(\text{Bd } N) \to G/G_\omega$$

is an epimorphism.

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2. NONRETRACTABLE CUBES-WITH-HOLES

THEOREM 1. *For each integer* $n \geq 2$, *there exists a nonretractable cube-with-holes of genus* n.

Consider the graph Γ having Euler characteristic -1 and embedded in S^3 as indicated in Figure 1. Let G denote the fundamental group of $S^3 - \Gamma$.

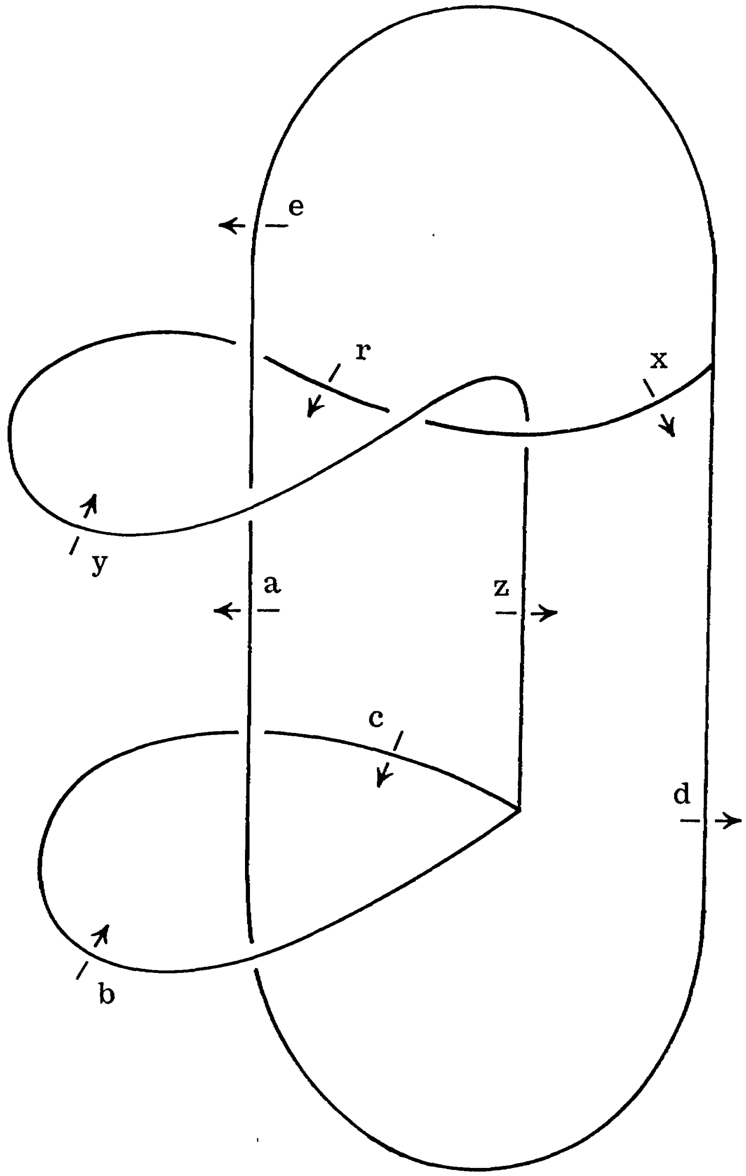


Figure 1.

The relations

$$c = a^{-1}ba, \qquad x = [b, a][x^{-1}, a^{-1}],$$

$$d = b^{-1}ab, \qquad y = x^{-1}[b, a]x,$$

$$e = (b^{-1}ab)x, \qquad z = [b, a],$$

$$r = [x, [a, b]]x$$

can be read from Figure 1. Hence, the group G has the presentation

$$G \equiv \{a, b, x: x = [b, a][x^{-1}, a^{-1}]\} .$$

Let N denote a regular neighborhood of Γ in S^3, and let $M_2 = S^3$ - Int N. Then M_2 is a cube-with-holes of genus 2. We shall show that M_2 is a nonretractable cube-with-holes of genus 2.

Let F_2 denote the free group of rank 2, and suppose $\{p, q\}$ is a set of free generators for F_2. Then the function ϕ that takes a to p, b to q, and x to 1 extends to a homomorphism (also called ϕ) of G onto $F_2/[F_2, F_2]$.

Let $T = S^1 \times S^1$ denote the two-dimensional torus. A mapping f of a space X into T is called *unstable* if it is homotopic to a mapping into a proper subset of T. Otherwise, f is *stable*. Let W denote the wedge $W = S^1 \vee S^1$ naturally contained in T. We can identify $\pi_1(W)$ with the free group F_2 in such a way that T is obtained from W by identification of a two-cell B^2 to W along Bd B^2 *via* the word [q, p].

Since $\pi_2(T) = 0$, there is a map $f: M_2 \to T$ such that $f_* = \phi$. However, we have the following theorem (which is Theorem 4 of [4]).

THEOREM. *Let* M *be a compact* 3-*manifold, possibly with boundary, and suppose* $H_1(M; Z)$ *is a free abelian group of rank* 2. *Then there exists a retraction of* M *onto a wedge of two simple closed curves if and only if every mapping of* M *into the torus* $T = S^1 \times S^1$ *is unstable.*

In particular, if M_2 were a retractable cube-with-holes, then the mapping $f: M_2 \to T$ would be unstable. We shall show that the assumption that f is unstable leads to a contradiction.

If f is unstable, then there exists a mapping $g: M \to W$ such that the diagram

$$\begin{array}{ccc} M_2 & \xrightarrow{f} & T \\ & {\scriptstyle g}\searrow \quad \nearrow{\scriptstyle i} & \\ & W & \end{array}$$

is homotopy-commutative (we use i to denote the inclusion mapping of W into T). It seems unclear whether the induced homomorphism $g_*: G \to F_2$ is always a surjective homomorphism. If it were, this would make life a bit easier.

We consider the subgroup F of F_2, where $F = g_*(G)$. The group F is free, and since $G/[G, G]$ is free Abelian of rank 2, the rank of F is at most 2.

LEMMA 1. *The rank of* F *is* 2.

Proof. Suppose the rank of F is less than 2. Since $i_* g_* = f_*$, there exist elements C_a, $C_b \in [F_2, F_2]$ such that $g_*(a) = pC_a$ and $g_*(b) = qC_b$. Clearly, $pC_a \neq 1 \neq qC_b$; therefore, the rank of F can only be 1. That is, F is infinite cyclic. Let z denote a generator of F. Then there exist integers j and k such that $pC_a = z^j$ and $qC_b = z^k$. It follows that $(pC_a)^k = (qC_b)^j$. Let $\bar{p}$ and $\bar{q}$ denote the equivalence classes determined by p and q in $F_2/[F_2, F_2]$. Then $(\bar{p})^k = (\bar{q})^j$. But $F_2/[F_2, F_2]$ is free Abelian on the elements $\bar{p}$ and $\bar{q}$. Arriving at this contradiction, we have proved Lemma 1.

If K is a group and $\{x_1, \cdots, x_k\}$ are elements in K, we denote the *normal closure* of $\{x_1, \cdots, x_k\}$ in K by $\langle x_1, \cdots, x_k \rangle$.

We now consider g_* as a homomorphism of G onto F. Let $u = g_*(a)$ and $v = g_*(b)$. We shall show that u and v are associated primitive elements of F. Consider the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{g_*} & F \\ \rho \downarrow & & \downarrow \tilde{\rho} \\ G/\langle [a, b] \rangle & \xrightarrow{\tilde{g}_*} & F/\langle [u, v] \rangle \end{array}$$

where ρ and $\tilde{\rho}$ are natural projections and $\tilde{g}_*$ is induced by g_*. The existence of $\tilde{g}_*$ follows from the inclusion relation

$$g_* \langle [a, b] \rangle \subseteq \langle [u, v] \rangle .$$

LEMMA 2. $G/\langle [a, b] \rangle$ *is Abelian.*

Proof. It is sufficient to show that the generators $\rho(a)$, $\rho(b)$, and $\rho(x)$ of $G/\langle [a, b] \rangle$ commute. The relation $[a, b] \in \langle [a, b] \rangle$ implies that $[\rho(a), \rho(b)] = 1$. Furthermore,

$$\rho(x) = \rho([b, a][x^{-1}, a^{-1}]) = \rho(x)\rho(a)\rho(x^{-1})\rho(a^{-1}) .$$

Hence, $\rho(a)\rho(x^{-1})\rho(a^{-1}) = 1$, and it follows that $\rho(x) = 1$. This completes the proof of Lemma 2.

Since g_* is surjective, the homomorphism $\tilde{g}_*$ is surjective and the quotient group $F/\langle [u, v] \rangle$ of F is Abelian. Let r and s be associated primitive elements for F. Then $\langle [u, v] \rangle \subseteq \langle [r, s] \rangle$, and the argument above shows that $\langle [r, s] \rangle \subseteq \langle [u, v] \rangle$. Therefore, $F/\langle [u, v] \rangle$ is free Abelian of rank 2. By the remark on page 266 of [6], $[u, v]$ is a conjugate of $[r, s]$ or $[s, r]$. By statement (11) on page 293 of [7], the elements u, v are associated primitive elements of F.

LEMMA 3. *There exists no element* $w \in F$ *such that* $w = [v, u][w^{-1}, u^{-1}]$.

Proof. Each word $\bar{w}$ in F has the form $\bar{w} = u^{\varepsilon_1} v^{\delta_1} \cdots u^{\varepsilon_n} v^{\delta_n}$, where $\varepsilon_i \neq 0$ $(1 < i \leq n)$ and $\delta_i \neq 0$ $(1 \leq i < n)$ are integers. If $\varepsilon_1 \neq 0 \neq \delta_n$, we say $\bar{w}$ has *length* $\ell(\bar{w}) = 2n$; if $\varepsilon_1 \neq 0$ and $\delta_n = 0$ or $\varepsilon_1 = 0$ and $\delta_n \neq 0$, $\ell(\bar{w}) = 2n - 1$; and if $\varepsilon_1 = 0 = \delta_n$, $\ell(\bar{w}) = 2(n - 1)$. The length of a word $\bar{w}$ is a well-defined function of $\bar{w}$.

Suppose that $w = u^{\varepsilon_1} v^{\delta_1} \cdots u^{\varepsilon_n} v^{\delta_n}$ is an element of F such that each ε_i and each δ_i satisfies the conditions above. Furthermore, suppose that

$$w = [v, u][w^{-1}, u^{-1}] .$$

Since $w \in [F, F]$, we see that $\ell(w) \geq 4$.

Case 1. If $\ell(w) = 2n$, then $n \geq 2$ and

$$w = (v^{-1} u^{-1} v)(u^{(\varepsilon_1 + 1)} v^{\delta_1} \cdots u^{\varepsilon_n} v^{\delta_n})(u v^{-\delta_n} \cdots v^{-\delta_1} u^{-(\varepsilon_1 + 1)}) .$$

If $\varepsilon_1 \neq -1$ or $n > 2$, we readily obtain a contradiction to the uniqueness of $\ell(w)$. Suppose, therefore, that $\varepsilon_1 = -1$ and $n = 2$. Then

$$w = (v^{-1}u^{-1}v)(v^{\delta_1}u^{\varepsilon_2}v^{\delta_2})(u v^{-\delta_2}u^{-\varepsilon_2}v^{-\delta_1}).$$

Cancellation is maximized if $\delta_1 = -1$, $\varepsilon_2 = 1$, and $\delta_2 = 1$. But in this case

$$u^{-1}v^{-1}uv = uv^{-1}u^{-1}v,$$

and this contradicts the fact that F is a free group on the elements u, v.

Case 2. If $\ell(w) = 2n - 1$ and $\varepsilon_1 \neq 0$, then $n > 2$ and

$$w = (v^{-1}u^{-1}v)(u^{(\varepsilon_1+1)}v^{\delta_1}\cdots v^{\delta_{n-1}}u)(v^{-\delta_{n-1}}\cdots v^{-\delta_1}u^{-(\varepsilon_1+1)}).$$

If $\varepsilon_1 \neq 1$, it is again easy to obtain a contradiction to the uniqueness of $\ell(w)$. Suppose, therefore, that $\varepsilon_1 = -1$. Then

$$w = (v^{-1}u^{-1}v)(v^{\delta_1}\cdots v^{\delta_{n-1}}u)(v^{-\delta_{n-1}}\cdots v^{-\delta_1}).$$

If $n > 3$, then the minimum length of the word on the right-hand side of the equation is 2n. If $n = 3$, then the cancellation is maximized if $\delta_1 = -1$, $\varepsilon_2 = 1$, and $\delta_2 = 1$. But in this case the length of the right-hand side of the equation is 4, whereas $\ell(w) = 5$, by hypothesis.

Case 3. If $\ell(w) = 2n - 1$ and $\delta_n \neq 0$, then $n > 2$ and

$$w = (v^{-1}u^{-1}vu)(v^{\delta_1}\cdots u^{\varepsilon_n}v^{\delta_n})u(v^{-\delta_n}u^{-\varepsilon_n}\cdots v^{-\delta_1})u^{-1}.$$

This gives an immediate contradiction.

Case 4. If $\ell(w) = 2(n - 1)$, then $n > 2$,

$$w = (v^{-1}u^{-1}vu)(v^{\delta_1}\cdots v^{\delta_{n-1}})u(v^{-\delta_{n-1}}\cdots v^{-\delta_1})u^{-1},$$

and the length of the right-hand side of the equation is at least 2n.

This completes the proof of Lemma 3.

If a homomorphism g_* of G onto F were to exist, with $g_*(x) = w$, then it would be necessary that

$$w = [v, u][w^{-1}, u^{-1}].$$

This contradiction completes the proof that M_2 is a nonretractable cube-with-holes of genus 2.

If K is a group, we define the *inner rank* of K to be the upper bound of the ranks of free homomorphic images of K. We denote the inner rank of a finitely generated group K by IN(K). The free product of the groups G_1 and G_2 is denoted by $G_1 * G_2$. The following is Theorem 3.2 of [3].

THEOREM. *Suppose* G_1 *and* G_2 *are finitely presented groups. Then*

$$\mathrm{IN}(G_1 * G_2) = \mathrm{IN}(G_1) + \mathrm{IN}(G_2).$$

Let M and N be orientable 3-manifolds with nonvoid boundary. Let D_M and D_N denote disks in Bd M and Bd N, respectively. Let $h\colon D_M \to D_N$ denote an orientation-reversing homeomorphism. The 3-manifold obtained by identification of D_M with D_N *via* the homeomorphism h is called *a disk sum* of M and N. We usually denote a disk sum of M and N by $M \Delta N$.

Suppose H_{n-2} is a cube-with-handles of genus $n - 2$ $(n \geq 2)$. Let $M_n = M_2 \Delta H_{n-2}$. Then M_n is a cube-with-holes of genus n. Clearly, such a disk sum $M_2 \Delta H_{n-2}$ can be embedded in S^3. However, by [1], any compact 3-manifold with connected boundary embedded in S^3 is a cube-with-holes.

Consider the cube-with-holes $M_n = M_2 \Delta H_{n-2}$, if $n \geq 3$. We have shown that M_2 is nonretractable. We shall show that M_n $(n \geq 3)$ is nonretractable.

Suppose M_n $(n \geq 3)$ were retractable. We have as a corollary to Theorem 2 of [4] the following result.

THEOREM. *Let* M *denote a cube-with-holes, and let* K *denote the fundamental group of* M. *Let* F *be a free group of rank* n. *Then there exists a homomorphism of* K *onto* F *if and only if there exists a retraction of* M *onto a wedge of* n *simple closed curves.*

Let $K = \pi_1(M_n)$; then $IN(K) = n$. However, $K \approx \pi_1(M_2) * \pi_1(H_{n-2})$. Since M_2 is nonretractable, $IN(\pi_1(M_2)) = 1$. The group $\pi_1(H_{n-2})$ is a free group of rank $n - 2$; hence, $IN(\pi_1(H_{n-2})) = n - 2$. Since inner rank is summable over a free product, $IN(K) = n - 1$. This contradiction completes the proof of Theorem 1.

3. RETRACTABLE AND BOUNDARY-RETRACTABLE CUBES-WITH-HOLES

Let M_n denote a cube-with-holes of genus n, and suppose H_n is a cube-with-handles of genus n. A mapping

$$f: (M_n, \text{Bd } M_n) \rightarrow (H_n, \text{Bd } H_n)$$

is said to be *boundary-preserving* if $f \mid \text{Bd } M_n$ maps $\text{Bd } M_n$ homeomorphically onto $\text{Bd } H_n$.

THEOREM 2. *Let* N *be a cube-with-holes, and let* G *denote the fundamental group of* N. *If* N *is boundary-retractable, then the natural map*

$$\pi_1(\text{Bd } N) \rightarrow G/G_\omega$$

is an epimorphism.

Proof. By Theorem 3, page 153 of [4] and the fact that N is boundary-retractable, there exists a boundary-preserving map f of N onto the cube-with-handles H. Hence $(f \mid \text{Bd } N)_*$ is an isomorphism of $\pi_1(\text{Bd } N)$ *onto* $\pi_1(\text{Bd } H)$. Now let g be a loop in G (based on Bd N). Choose $\gamma \in \pi_1(\text{Bd } H)$ so that γ and $f_*(g)$ are equivalent as elements in $\pi_1(H)$. This is possible, since the inclusion of Bd H into H induces a homomorphism of $\pi_1(\text{Bd } H)$ *onto* $\pi_1(H)$. Hence, there exists a loop $\ell \in \pi_1(\text{Bd } N)$ such that $f_*(\ell)$ and γ are equivalent as elements in $\pi_1(\text{Bd } H)$. It follows that $f_*(g\ell^{-1})$ is equivalent to 1 in $\pi_1(H)$.

Now f_* is an epimorphism; therefore, by the corollary to Theorem 1 on page 151 of [4], $\ker f_* = G_\omega$. That is, the class of $g\ell^{-1}$ is an element in G_ω.

Let T denote a wedge at t_0 of n simple closed curves $T_1, \cdots, T_n$. Suppose t is a point of T such that $t \neq t_0$. A PL map f of the compact 3-manifold M into T is said to be *transverse with respect to* t if it satisfies the following two conditions.

1) Each component of $f^{-1}(t)$ is a properly embedded, polyhedral surface in M.

2) If S is a component of $f^{-1}(t)$, then there exist a closed neighborhood U(t) of t in $T - t_0$, a homeomorphism ϕ: $t \times [-1, 1]$ onto U(t), and a homeomorphism ψ of $S \times [-1, 1]$ onto the component U(S) of $f^{-1}(U(t))$ containing S such that f maps each arc $\psi(s \times [-1, 1])$ homeomorphically onto U(t), satisfying the equation $f\psi(s, r) = \phi(t, r)$ $(-1 \leq r \leq 1)$.

Suppose $\{t_1, \cdots, t_p\}$ is a collection of points in $T - \{t_0\}$. The PL map f of a compact 3-manifold M into T is said to be *transverse with respect to* $\{t_1, \cdots, t_p\}$ if f is transverse with respect to each t_i $(1 \leq i \leq p)$.

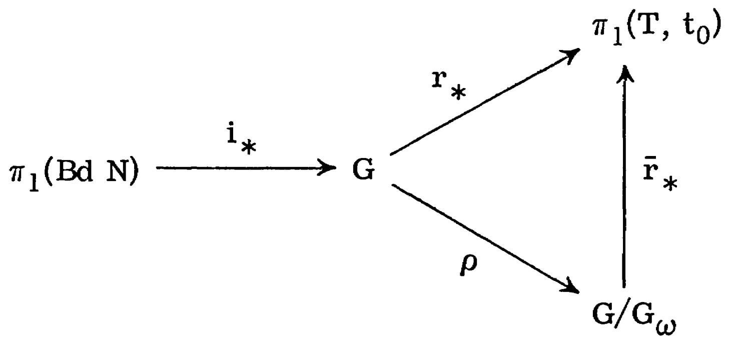
The following theorem constitutes a partial converse of Theorem 2.

THEOREM 3. *Let* N *be a cube-with-holes, and let* G *denote the fundamental group of* N. *If* N *is retractable and the natural map*

$$\pi_1 (\mathrm{Bd}\ N) \to G/G_\omega$$

is an epimorphism, then N *is boundary-retractable.*

Proof. Suppose $n \geq 1$ is the genus of N. Let T be a wedge at t_0 of n polyhedral, simple closed curves $T_1, \cdots, T_n$ in Int N such that there exists a PL retraction r of N onto T. By the corollary to Theorem 1 on page 151 of [4], the factor group G/G_ω is isomorphic to the fundamental group of T, which is free of rank n. Let i_* denote the homomorphism of $\pi_1(\mathrm{Bd}\ N)$ into G induced by inclusion. If ρ is the natural projection of G onto G/G_ω and $\bar{r}_*$ is the isomorphism of G/G_ω onto $\pi_1(T, t_0)$ induced by r_*, then the diagram



commutes. Since $\rho \circ i_*$ is an epimorphism and $\bar{r}_*$ is an isomorphism of G/G_ω *onto* $\pi_1(T, t_0)$, we see that $r_* \circ i_*$ is an epimorphism. The homomorphism $r_* \circ i_*$ is identical to $(r \mid \mathrm{Bd}\ N)_*$.

Choose a subdivision of T for which r is simplicial. For $1 \leq i \leq n$, choose a point $t_i \in T_i - \{t_0\}$ that is not a vertex in this subdivision of T. Then r is transverse with respect to $\{t_1, \cdots, t_n\}$. Let F_i denote the component of $r^{-1}(t_i)$ containing t_i. Each F_i is a polyhedral, regularly embedded, two-sided surface in N, and $N - \bigcup_{i=1}^n F_i$ is connected.

Each F_i meets Bd N, since each closed surface in Int N separates N. If it were true that each F_i met Bd N in precisely one simple closed curve, the proof would proceed as follows: let $J_i = \mathrm{Bd}\ F_i$. Then $\bigcup_{i=1}^n J_i$ does not separate Bd N. Hence, there exists a wedge B at b_0 of n polyhedral simple closed curves $B_1, \cdots, B_n$ in Bd N such that $B_i \cap J_i = \{b_i\}$ consists of exactly one crossing point and $B_i \cap J_j = \emptyset$ $(i \neq j)$. Let $U(F_i)$ $(1 \leq i \leq n)$ denote the interior of a small product neighborhood of F_i in N. If $U(F_i)$ is properly chosen, then $B - U(F_i)$ is a tree.

Hence, the projection of F_i onto b_i $(1 \leq i \leq n)$ may be extended to a retraction of N onto B. This construction is like that used in Theorem 2 on page 151 of [4].

To finish the proof of Theorem 3, we shall show that we can choose a retraction r of N onto T such that each F_i meets Bd N in precisely one simple closed curve. That is, there exist a retraction r of N onto T and a collection $\{t_1, \cdots, t_n\}$ of points of T such that $t_i \in T_i - t_0$, r is transverse with respect to $\{t_1, \cdots, t_n\}$, and such that if F_i is the component of $r^{-1}(t_i)$ containing t_i, then $F_i \cap \text{Bd } N = J_i$ is precisely one simple closed curve.

To this end, suppose that a PL retraction of N onto T is given and $t_i \in T_i - t_0$ is not a vertex point of a subdivision of T for which f is simplicial. Let L_i be the component of $f^{-1}(t_i)$ that contains t_i. Let $c(L_i)$ be one less than the number of components of $L_i \cap \text{Bd } N$. Then $c(L_i) \geq 0$. If $\sum_{i=1}^n c(L_i) = 0$, let $r = f$ and $F_i = L_i$. We shall show that if $\sum_{i=1}^n c(L_i) = k > 0$, then there exists a PL retraction f' of N onto T such that

(i) f' is homotopic to f (Rel $\{t_0\}$),

(ii) if L_i' denotes the component of $(f')^{-1}(t_i)$ containing t_i and if $c(L_i')$ is one less than the number of components of $L_i' \cap \text{Bd } N$, then $\sum_{i=1}^n c(L_i') = k - 1$, and

(iii) f' is transverse with respect to $\{t_1, \cdots, t_n\}$.

Since $\sum_{i=1}^n c(L_i) > 0$, there is an argument similar to that in the proof of Lemma 3 on page 369 of [2] to find a j $(1 \leq j \leq n)$ such that $L_j \cap \text{Bd } N$ has distinct components J_0 and J_1; such that there is an arc A in Bd N from J_0 to J_1 with $A \cap \bigcup_{i=1}^n L_i = \text{Bd } A$; and such that $f \mid A$ is a homotopically trivial loop in T based at t_j.

Let Q_0 and Q_1 be small disjoint disks in $L_j - \{t_j\}$, chosen so that for $m = 0, 1$, the set $Q_m \cap \text{Bd } N = A_m'$ is a small arc in J_m having an end point of A in its interior. Let $Q \subset N - T$ be a regular neighborhood of A, chosen so that

$$Q \cap \bigcup_{i=1}^{n} L_i = Q_0 \cup Q_1 \subset \text{Bd } Q,$$

and so that

$$Q \cap \text{Bd } N = D'$$

is a disk in Bd Q for which A is a spanning arc. Then the closure of

$$\text{Bd } Q - (Q_0 \cup Q_1 \cup D')$$

is a disk D. The disk D meets Bd N in the disjoint arcs A_0 and A_1 from J_0 to J_1. A slight modification of Lemma 3.1 on page 361 of [2] yields a retraction f' of N onto T such that

(i) f' is homotopic to f (Rel $\{T\}$),

(ii) the component of $(f')^{-1}(t_i)$ containing t_i is L_i $(i \neq j)$,

(iii) the component of $(f')^{-1}(t_j)$ containing t_j is $L_j \cup D - (Q_0 \cup Q_1)$, and

(iv) f' is transverse with respect to $\{t_1, \cdots, t_n\}$.

Hence, if L_i' denotes the component of $(f')^{-1}(t_i)$ containing t_i, then $c(L_i') = c(L_i)$ $(i \neq j)$, and $c(L_j') = c(L_j) - 1$. This is true since the distinct components J_0 and J_1 of $L_j \cap \text{Bd } N$ have been altered to a single component

$$(J_0 \cup J_1) \cup (A_0 \cup A_1) - (A_0' \cup A_1').$$

This completes the proof of Theorem 3.

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