THE COVERING THEOREM FOR UPPER BASIC SUBGROUPS

Paul Hill

All groups considered in this paper are primary abelian groups. We follow the notation and terminology of [1]. Recall that a subgroup B of G is a *basic subgroup* of G if it has the following properties:

(i) B is a direct sum of cyclic groups,

(ii) B is pure in G,

(iii) G/B is divisible.

Let r(G) denote the rank of a group G, and set

$$r_G = \min\{r(G/B)\colon B \text{ is a basic subgroup of } G\}.$$

If B is a basic subgroup of G such that $r(G/B) = r_G$, then B is called an *upper* basic subgroup of G. As L. Fuchs has mentioned in [1, p. 105], upper basic subgroups are important because an upper basic subgroup B of G, together with r(G/B), may reveal much more information about the structure of G than can be conveyed by an arbitrary basic subgroup.

A. R. Mitchell [6] has proved the following for reduced p-groups. If B_1 and B_2 are upper basic subgroups of G_1 and G_2, respectively, then $B_1 + B_2$ is an upper basic subgroup of $G_1 + G_2$. If B is an upper basic subgroup of a high subgroup H of G, then B is also an upper basic subgroup of G. Left open in [6] was the question whether each basic subgroup is contained in an upper basic subgroup. In his review of Mitchell's paper, in the *Zentralblatt* (166, p. 292), P. Grosse stated "whereas the author could not prove that a basic subgroup B of G is contained in an upper basic subgroup whenever G is a reduced p-group he paved the way to this (hopefully correct) statement." In this paper we settle the question affirmatively, but travel a different (unpaved) route. The solution comes as an immediate corollary to a structure theorem (Theorem 3) that gives almost complete information about basic subgroups that are not upper in relation to upper basic subgroups. The main result of the paper, however, is the following. *If* G *is a direct sum of cyclic groups and if* B *and* B' *are basic subgroups of* G *such that* $G/B \cong G/B'$, *then there exists an automorphism* π *of* G *such that* $\pi(B) = B'$. We cast this result in slightly more general form (Theorem 1). The proof involves extending height-preserving automorphisms on subgroups; for related results, see [2] and [3].

THEOREM 1. *Let* G *be a primary group, and let* $G = G_0 + H$, *where* H *is a direct sum of cyclic groups. Suppose that* B_0 *is a basic subgroup of* G_0. *Let* $B = B_0 + A$ *and* $B' = B_0 + A'$ *be basic subgroups of* G. *There exists an automorphism* π *of* G *that is the identity on* G_0 *and maps* B *onto* B' *if and only if*

(I) $$G/\{G_0, B\} = G/\{G_0, B'\}.$$

Received October 17, 1969.
This research was supported by NSF Grant GP-12818.

Michigan Math. J. 18 (1971).

Proof. Condition (I) is obviously necessary, but its sufficiency is not trivial. We shall begin the proof with some preliminary remarks. Both A and A' are isomorphic, in a natural way, to basic subgroups of H. Therefore $A \cong H \cong A'$, since H is a direct sum of cyclic groups. Write

$$H = \sum_I \{h_i\}, \qquad A = \sum_I \{a_i\}, \qquad A' = \sum_I \{a_i'\}.$$

Note that $G/\{G_0, B\}$ is divisible, since G/B is divisible. Thus we can write

$$G/\{G_0, B\} = \sum_J D_j \quad \text{and} \quad G/\{G_0, B'\} = \sum_J D_j',$$

where $D_j \cong Z(p^\infty) \cong D_j'$ for each j; in view of condition (I), we can use a common index set J. Observe that $\{G_0, B\}$ and $\{G_0, B'\}$ are pure subgroups of G. It is convenient to choose the index sets I and J to be the smallest possible ordinals. Thus we let

$$I = \{\alpha: \alpha \text{ is an ordinal of cardinality less than } r(H)\},$$

$$J = \{\alpha: \alpha \text{ is an ordinal of cardinality less than } r(G/\{G_0, B\})\}.$$

In this connection, we remark that the theorem follows immediately if either I or J is vacuous. If J is vacuous, then $G_0 + A = G = G_0 + A'$, and we obtain the desired automorphism of G by mapping A isomorphically onto A'. Observe that $J \subseteq I$ (compare $H \cong G/G_0$ to $G/\{G_0, B\}$). If I is finite, then $G_0 + A = G = G_0 + A'$, since a finite group cannot have a proper basic subgroup; for notational convenience we therefore assume that I is infinite.

The proof of the theorem is a two-stage induction proof. The first stage of the induction is the following. Let π_0 denote the identity map of G_0. Suppose that $\gamma \in I$ and that for each $\alpha < \gamma$, we have a subgroup G_α of G and a height-preserving automorphism π_α of G_α (all heights are computed with respect to G) such that $G_\alpha \subseteq G_\beta$ and π_β is an extension of π_α if $\alpha < \beta < \gamma$. Assume, for each $\alpha < \gamma$, that $\pi_\alpha(G_\alpha \cap B) = G_\alpha \cap B'$. Further, assume that the following conditions are satisfied for suitable subsets $I(\alpha)$ of I and $J(\alpha)$ of J.

(1) $$G_\alpha = G_0 + \sum_{I(\alpha)} \{h_i\},$$

(2) $$G_\alpha \cap B = B_0 + \sum_{I(\alpha)} \{a_i\},$$

(3) $$G_\alpha \cap B' = B_0 + \sum_{I(\alpha)} \{a_i'\},$$

(4) $$\{G_\alpha, B\}/\{G_0, B\} = \sum_{J(\alpha)} D_j,$$

(5) $$\{G_\alpha, B'\}/\{G_0, B'\} = \sum_{J(\alpha)} D_j'.$$

If γ is a limit ordinal, we define $G_\gamma = \bigcup_{\alpha<\gamma} G_\alpha$, and we let π_γ denote the automorphism of G_γ such that $\pi_\gamma | G_\alpha = \pi_\alpha$ for each $\alpha < \gamma$. The induction hypotheses continue to hold for $\alpha < \gamma$ if we let $I(\gamma) = \bigcup_{\alpha<\gamma} I(\alpha)$ and $J(\gamma) = \bigcup_{\alpha<\gamma} J(\alpha)$. The nontrivial case arises when $\gamma - 1$ exists, and here we employ the second stage of the induction to pass from $\gamma - 1$ to γ. Of course, we could keep the induction technically alive by taking $G_\gamma = G_{\gamma-1}$ and $\pi_\gamma = \pi_{\gamma-1}$, but there is no point in that. For we want the subgroups G_α eventually to exhaust G. Thus it is appropriate to add the following condition to the induction hypotheses:

(6) If $\alpha + 1 < \gamma$, then $\alpha \in I(\alpha + 1)$.

For notational convenience in working with the secondary induction, we write $\beta = \gamma - 1$.

Assume that K and L are finite extensions of G_β and that ϕ: K $\rightarrowtail\!\!\twoheadrightarrow$ L is a height-preserving isomorphism from K onto L such that

(*) $$\phi(K \cap B) = L \cap B'.$$

We need to extend ϕ to a height-preserving automorphism π_γ of a subgroup G_γ containing both K and L such that conditions (1) to (6) hold for $\alpha \leq \gamma$ and such that $\pi_\gamma(G_\gamma \cap B) = G_\gamma \cap B'$. In particular, we want β to belong to I_γ. Let $x \in G$. We claim that there exist finite extensions K^+ and L^+ of K and L (both containing x) and a height-preserving isomorphism ϕ^+: $K^+ \rightarrowtail\!\!\twoheadrightarrow L^+$ that extends ϕ, with the additional property $\phi^+(K^+ \cap B) = L^+ \cap B'$. Because of the symmetry of K and L, it suffices to show that we can do this with x in K^+ (but not necessarily in L^+); by a second application we can put x into L^+. Thus we want to extend ϕ to a height-preserving isomorphism ϕ^+ from $K^+ = \{K, x\}$ into G such that

$$\phi^+(K^+ \cap B) = \phi^+(K^+) \cap B'.$$

Without loss of generality, we assume that the element x is not in K, but that px is in K, and that among the elements of the coset x + K, the element x has maximal height in G. (Such an element is called a *proper element* with respect to K.) There is no problem at all in exchanging representatives of the coset x + K, but we must verify that the coset x + K does indeed have an element of greatest height; it does because K is a finite extension of a direct summand G_β of G. Let h(x) denote the height of x, and let h(x) = n. If $h(p(x+k)) > n+1$ for some $k \in K \cap p^nG$, we assume that $h(px) > n+1$; in other words, if among the elements of the coset x + K that are proper with respect to K there exists y such that $h(py) > n+1$, then we choose x to have this property. First we shall consider the other case.

Case 1. $h(px) = n+1$. There are two subcases.

Subcase A. $x + k \in B$ for some $k \in K$. Choose $y_0 \in p^nG$ so that $\phi(px) = py_0$; this is possible, since ϕ preserves heights. If $y_0 + \phi(k) \in B'$, set $y = y_0$. If $y_0 + \phi(k) \notin B'$, observe that $p(y_0 + \phi(k)) = \phi(p(x+k)) \in pB'$, and let $y_0 + \phi(k) = b' + t$, where $b' \in B'$ and $pt = 0$. Since B' is a basic subgroup of G, we can write $t = c' + z$, where $c' \in B'[p]$ and $z \in p^{n+1}G[p]$. Set $y = y_0 - z$. Then $y + \phi(k) \in B'$. Extend ϕ by mapping x onto y.

Subcase B. $x + k \in B$ for no $k \in K$. Choose $y_0 \in p^nG$ so that $\phi(px) = py_0$. If $y_0 + \phi(k) \notin B'$ for all $k \in K$, let $y = y_0$. If $y_0 + \phi(k) \in B'$ for some $k \in K$, consider $p(x+k)$. The homomorphism ϕ maps $p(x+k)$ onto $p(y_0 + \phi(k)) \in pB'$. Hence

$p(x+k) \in pB$. Write $x+k = b+s$, where $b \in B$ and $ps = 0$. Now $s \notin \{B, K\}$; otherwise, Subcase *B* would be voided. We claim that there exists $t \in G[p]$ such that $t \notin \{B', L\}$. First, notice that this is certainly the case if $K = G_\beta$ (= L), for by the induction hypothesis, $G/\{B, G_\beta\} \cong G/\{B', G_\beta\}$. Since $\{B, G_\beta\}$ and $\{B', G_\beta\}$ are pure subgroups of G, it follows that

$$\{G[p], B, G_\beta\}/\{B, G_\beta\} \cong \{G[p], B', G_\beta\}/\{B', G_\beta\}.$$

The isomorphism $\phi: K \rightarrowtail\!\!\twoheadrightarrow L$ induces an isomorphism

$$\{B, K\}/\{B, G_\beta\} \rightarrowtail\!\!\twoheadrightarrow \{B', L\}/\{B', G_\beta\}.$$

Thus

$$\{G[p], B, G_\beta\}/\{\{B, K\}[p], B, G_\beta\} \cong \{G[p], B', G_\beta\}/\{B', L\}[p], B', G_\beta\},$$

and since the left-hand side is not zero, it follows that

$$\{G[p], B', G_\beta\} \neq \{\{B', L\}[p], B', G_\beta\}.$$

Therefore, there exists $t \in G[p]$ such that $t \notin \{B', L\}$. Choose $b' \in B'[p]$ so that $t - b' \in p^{n+1}G$. Define $y = y_0 + t - b'$, and note that $y \notin \{B', L\}$. Thus $y + \phi(k) \notin B'$ for all $k \in K$. Extend ϕ by mapping x onto y.

Case 2. $h(px) > n+1$. Again, there are two subcases.

Subcase A. $x+k \in B$ for some $k \in K$. Choose $w \in p^{n+1}G$ so that $pw = \phi(px)$. By the usual argument, some $z \in G[p]$ is proper with respect to L and has height exactly n. Set $y_0 = w + z$. If $y_0 + \phi(k) \in B'$, let $y = y_0$. If $y_0 + \phi(k) \notin B'$, define y in terms of y_0 exactly as in Case 1, Subcase *A*. Extend ϕ by mapping x onto y.

Subcase B. $x+k \notin B$ for all $k \in K$. Define y_0 in the same way as in the preceding case, $y_0 = w + z$. If $y_0 + \phi(k) \in B'$ for no $k \in K$, let $y = y_0$. If $y_0 + \phi(k) \in B$ for some $k \in K$, observe that $\phi(p(x+k)) \in pB'$. Hence $p(x+k) \in pB$. Write $x+k = b+s$, where $b \in B$ and $ps = 0$. Since $s \notin \{K, B\}$, there exists $t \in G[p]$ such that $t \notin \{L, B'\}$; the argument is the same as in Case 1, Subcase *B*. Again, we let $y = y_0 + t - b'$, where $b' \in B'[p]$ and $t - b' \in p^{n+1}G[p]$. Extend ϕ by mapping x onto y.

In all four cases, the extension ϕ^+ of ϕ to $K^+ = \{K, x\}$ is a height-preserving isomorphism from K^+ into G such that $\phi^+(K^+ \cap B) = \phi^+(K^+) \cap B'$. This completes the secondary induction, and it is actually all we need to complete the first-stage induction. The argument that this suffices to finish the first-stage induction is outlined below.

Since each of the summands in the $\sum$-summations involved in conditions (1) to (5) is either finite or countably infinite, we can take ascending sequences

$$G_\beta = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n \subseteq \cdots$$

and

$$G_\beta = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n \subseteq \cdots$$

of finite extensions of G_β such that $\bigcup K_n = \bigcup L_n$ and such that there exists a sequence $\phi_n: K_n \rightarrowtail\!\!\twoheadrightarrow L_n$ of height-preserving isomorphisms that also ascend

(ϕ_{n+1} extends ϕ_n) and have the property $\phi_n(K_n \cap B) = L_n \cap B'$. Furthermore, if we let $G_\gamma = \bigcup K_n = \bigcup L_n$, then we can choose the groups K_n and L_n so that conditions (1) to (5) hold for $\alpha = \gamma$, where $I(\gamma)$ and $J(\gamma)$ are countable extensions of $I(\beta)$ and $J(\beta)$, respectively. (For a discussion of the details on how to do this, see [5]; the technique involved is sometimes referred to as the back-and-forth method of Hill and Megibben.) We can meet the final requirement, condition (6), by letting $h \in K_1 \cap L_1$, for then condition (1) implies that $\beta \in I(\gamma)$. Let $\pi_\gamma = \sup\{\phi_n\}$. Since we have retained the important condition $\pi_\gamma(G_\gamma \cap B) = G_\gamma \cap B'$, the first-stage induction goes through, and we thereby obtain an automorphism of G that maps B onto B'.

COROLLARY 2. *Let* G *be a primary group, and let* $G = G_0 + H$, *where* H *is a direct sum of cyclic groups. Suppose that* B *is a basic subgroup of* G *such that* $B \cap G_0$ *is a basic subgroup of* G_0. *Then there exists a decomposition* $G = G_0 + K$ *of* G *such that* $B = (B \cap G_0) + (B \cap K)$.

Proof. Let $B_0 = B \cap G_0$. Then B/B_0 is isomorphic to a subgroup of H. Since H is a direct sum of cyclic groups, B/B_0 is a direct sum of cyclic groups, and since B_0 is pure, it follows that $B = B_0 + A$ for some $A \subseteq B$. Since

$$B/B_0 \subseteq G/B_0 \cong (G_0/B_0) + H,$$

it follows that B/B_0 is isomorphic to a basic subgroup H_0 of H such that $G/(B_0 + H_0) \cong G/B$. By Theorem 1, some automorphism π of G maps $B_0 + H_0$ onto $B = B_0 + A$ and is the identity on G_0. Setting $K = \pi(H)$, we have the relation $G = G_0 + K$. Setting $C = \pi(H_0)$, we find that $B = B_0 + C$. Since $C \subseteq K$, the corollary follows.

Remark. We wish to acknowledge with appreciation an observation made by the referee. Corollary 2 is an easily proved proposition *without* Theorem 1; therefore Theorem 1 is not essential for the proof of the covering theorem (but it has applications elsewhere).

THEOREM 3. *If* G *is a* p-*group and* B *is a basic subgroup of* G, *then*

$$G = G_0 + K \quad \textit{and} \quad B = (B \cap G_0) + (B \cap K),$$

where $B \cap G_0$ *is an upper basic subgroup of* G_0 *and* K *is a direct sum of cyclic groups.*

Proof. Let $r(B/B_u) = m$, where B_u is an upper basic subgroup of G. Let B be an arbitrary basic subgroup of G. Then $G = G_0 + H$, where H is a direct sum of cyclic groups, $|G_0| \leq \max\{\aleph_0, m\}$, and $B \cap G_0$ is a basic subgroup of G_0; the argument is a simple application of the back-and-forth method. According to Corollary 2, there exists a decomposition $G = G_0 + K$ such that $B = (B \cap G_0) + (B \cap K)$. If $|G_0| = m$, then each basic subgroup of G_0 (in particular, $B \cap G_0$) is obviously an upper subgroup, for K is a direct sum of cyclic groups and $r(G/B_u) = m$. Thus the theorem is proved, except for the case where m is finite. In this case, G_0 is countable. Suppose that $r(G_0/(B \cap G_0)) = n \geq m$. It follows from [4] that $G_0 = G_0' + L$, where $B \cap G_0 = (B \cap G_0') + (B \cap L)$ and L is a direct sum of cyclic groups and $r(G_0'/(B \cap G_0') = m$. Write $G = G_0' + (L + K)$. Then $B \cap G_0'$ is an upper basic subgroup of G_0'. Since $B = (B \cap G_0') + B \cap (L + K)$, the theorem is proved.

COROLLARY 4. *Let* G *be a primary group. Each basic subgroup of* G *is contained in an upper basic subgroup of* G.

Proof. Let B be a basic subgroup of G, and let $G = G_0 + K$, where $B \cap G_0$ is an upper basic subgroup of G_0 and K is a direct sum of cyclic groups such that $B = (B \cap G_0) + (B \cap K)$. Then $(B \cap G_0) + K$ is an upper basic subgroup of G containing B.

REFERENCES

1. L. Fuchs, *Abelian groups*. Pergamon Press, New York, 1960.
2. P. Hill, *Extending automorphisms on primary groups*. Bull. Amer. Math. Soc. 74 (1968), 1123-1124.
3. ———, *Endomorphism rings generated by units*. Trans. Amer. Math. Soc. 141 (1969), 99-105.
4. ———, *Automorphisms of countable primary abelian groups*. Proc. Amer. Math. Soc. 25 (1970), 135-140.
5. P. Hill and C. Megibben, *Extending automorphisms and lifting decompositions in Abelian groups*. Math. Ann. 175 (1968), 159-168.
6. A. R. Mitchell, *Some properties of upper basic subgroups*. Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 10 (1967), 3-11.

Florida State University
Tallahassee, Florida 32306