ON THE COMPLETENESS OF NULLITY FOLIATIONS

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Several authors have studied the null-space distribution of intrinsic or extrinsic curvature tensors on a Riemannian manifold M (see the references below). That these distributions are integrable and their integral manifolds are totally geodesic is an easy consequence of Bianchi's identity or of the equation of Codazzi and Mainardi, respectively. In several cases, the integral manifolds have been shown to be complete if M is complete (see [1], [5], and [6]). On the other hand, in the study of curvature-like tensor fields or Riemannian double forms, A. Gray ([3] and [4]) assumed the curvature tensors to be recurrent, in order to obtain completeness. It is the aim of the present paper to eliminate this rather strong condition, and at the same time to simplify the previously known proofs of the completeness of the k-nullity and relative-nullity foliations.

The notion of Riemannian double forms with values in a vector bundle seems to be the proper setting for a simultaneous treatment of the extrinsic and the intrinsic case. We refer to [1], [2], and [4] for examples and geometric applications.

Notation and Assumptions. Let $(M, \langle \cdots, \cdots \rangle)$ be a Riemannian manifold with Levi-Civita covariant derivative $\overset{M}{\nabla}$, and let ξ be a vector bundle over M with covariant derivative $\overset{\xi}{\nabla}$. For each integer $p \geq 0$ and each vector bundle η over M, let $\Lambda^p(\eta)$ denote the bundle of alternating p-forms on M with values in η. We are particularly interested in the bundle $\delta_{p,q}(\xi) = \Lambda^p(\Lambda^q(\xi))$, the bundle of *double forms of type* (p, q) *with values in* ξ. Note that in the usual way $\overset{M}{\nabla}$ and $\overset{\xi}{\nabla}$ induce a covariant derivative $\overset{p,q}{\nabla}$ for $\delta_{p,q}(\xi)$. From now on we shall denote all occurring covariant derivatives simply by ∇.

For a double form $A \in \Gamma\delta_{p,q}(\xi)$, define $A^* \in \Gamma\delta_{p+1,q-1}(\xi)$ and $DA \in \Gamma\delta_{p+1,q}(\xi)$ by the equations

$$A^*(X_0, \cdots, X_p)(Y_2, \cdots, Y_q) = \sum_{j=0}^{p} (-1)^j A(X_0, \cdots, \hat{X}_j, \cdots, X_p)(X_j, Y_2, \cdots, Y_p)$$

and

$$(DA)(X_0, \cdots, X_p) = \sum_{j=0}^{p} (-1)^j (\nabla_{X_j} A)(X_0, \cdots, \hat{X}_j, \cdots, X_p),$$

for all vector fields $X_0, \cdots, X_p, Y_2, \cdots, Y_p$ on M.

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Definition. $A \in \Gamma\delta_{p,p}(\xi)$ is a *Riemannian double form* (*of type* (p, p)) *with values in* ξ if

$$A^* = 0, \tag{1}$$

$$DA = 0, \tag{2}$$

$$A(X_1, \cdots, X_p)(Y_1, \cdots, Y_p) = A(Y_1, \cdots, Y_p)(X_1, \cdots, X_p), \tag{3}$$

for all vector fields $X_1, \cdots, X_p, Y_1, \cdots, Y_p$ on M. As mentioned in [4], the symmetry condition (3) is already a consequence of (1).

Examples. 1. Let ξ be the trivial line bundle over M with its canonical connection, and let R be the Riemannian curvature tensor of M. Then

$$A(X_1, X_2)(Y_1, Y_2) = \langle R(X_1, X_2)Y_1, Y_2 \rangle$$

defines a Riemannian double form of type (2,2). The equations (1) and (2) are equivalent with the first and second Bianchi identity, respectively. Many other interesting examples of this form may be found in [4].

2. Let $f: M \to \hat{M}$ be an isometric immersion of M into a Riemannian manifold $\hat{M}$ of constant curvature. Let ξ be the normal bundle endowed with the normal connection. Then the second fundamental form of f is a Riemannian double form of type (1, 1) with values in ξ. Condition (1) simply indicates its symmetry, while (2) is equivalent with the equation of Codazzi and Mainardi. It is this example that led us to consider vector-valued forms.

Completeness. Let A be a Riemannian double form of type (p, p) with values in ξ. For $x \in M$, define

$$T_0(x) = \{X_1 \in M_x \mid A(X_1, \cdots, X_p) = 0 \text{ for all } X_2, \cdots, X_p \in M_x\}.$$

$T_0(x)$ is called the *nullity space of* A *at* x, and its dimension $\mu(x)$ is the *index of nullity of* A *at* x. It is easy to show that μ is upper-semicontinuous, and hence the set M_0 where μ attains its minimum is an open subset of M. Let τ_0 denote the subbundle of the tangent bundle $\tau(M_0)$ defined by $T_0 \mid M_0$.

THEOREM. *The bundle* τ_0 *is integrable, and its integral manifolds are totally geodesic. If* M *is complete, then the maximal integral manifolds of* τ_0 *are also complete.*

Proof. The proof of the first statements is verbally the same as given in [4] for the case where ξ is the trivial line bundle. We therefore confine ourselves to the proof of the completeness. Let L be one of the maximal integral manifolds of τ_0, and let $\gamma: [a, b] \to M$ be a unit-speed geodesic such that $\gamma([a, b[) \subset L$. We shall show that parallel translation along γ maps $T_0(\gamma(a))$ surjectively onto $T_0(\gamma(b))$. Then, obviously, $\gamma(b) \in M_0$, from which we conclude that the image of the extension of γ to $[a, \infty[$ is contained in L. But this proves the completeness of L.

Let $P: \tau(M_0) \to \tau_0^{\perp}$ denote the projection onto the orthogonal complement of τ_0. We define a cross section C in $\mathrm{Hom}(\tau_0, \mathrm{End}\,\tau_0^{\perp})$ by

$$C_U Y = -P(\nabla_Y U) \quad \text{for each } U \in \Gamma\tau_0 \text{ and each } Y \in \Gamma\tau_0^{\perp}.$$

Let $(\cdots)'$ denote the covariant derivative with respect to $\frac{d}{dt}$ of fields along γ, and set $\gamma_0 = \gamma | [a, b[$. Then it follows from the structural equations that for the tangent vector field $X = \gamma_* \frac{d}{dt}$ of γ, we have the relation

$$(4) \qquad (C_X)' = (C_X)^2 - P(R(X, \cdots)X)$$

on $[a, b[$ (see [2]). Here R is the Riemannian curvature tensor of M. Equation (4) implies that every $Z \in \Gamma\gamma_0^* \tau_0^\perp$ that satisfies the equation

$$(5) \qquad Z' + C_X Z = 0$$

is also a solution of the linear differential equation

$$(6) \qquad Z'' = P(R(X, Z)X)$$

and hence can be continued to a differentiable vector field along γ, on $[a, b]$. (Note that since L is totally geodesic, the bundle $\gamma_0^* \tau_0^\perp$ has a smooth extension to $[a, b]$ in which (6) makes sense.)

Now, given $X_0 \in \Gamma\tau_0$ and $X_1, \cdots, X_p \in \Gamma\tau_0^\perp$, we obtain from (2) the equation

$$0 = \sum_{j=0}^{p} (-1)^j (\nabla_{X_j} A)(X_0, \cdots, \hat{X}_j, \cdots, X_p)$$

$$= (\nabla_{X_0} A)(X_1, \cdots, X_p) - \sum_{j=1}^{p} (-1)^j A(\nabla_{X_j} X_0, X_1, \cdots, \hat{X}_j, \cdots, X_p)$$

$$= \nabla_{X_0}(A(X_1, \cdots, X_p)) - \sum_{j=1}^{p} A(X_1, \cdots, \nabla_{X_0} X_j + C_{X_0} X_j, \cdots, X_p) .$$

Hence, by extending vector fields along γ_0 locally to open sets in M_0, we obtain for each system $Z_1, \cdots, Z_p$ of solutions of (5) the equation

$$(7) \qquad (A(Z_1, \cdots, Z_p))' = 0 .$$

Now extend $Z_1, \cdots, Z_p$ differentiably to b, and let $Y_1, \cdots, Y_p$ be parallel vector fields along γ with $Y_1(b) \in T_0(\gamma(b))$. Then, according to (3),

$$A(Z_1, \cdots, Z_p)(Y_1, \cdots, Y_p)_t = 0$$

at $t = b$. From (7) it follows that this holds for $t = a$ as well. But since we can solve (5) with any initial $Z_1(a), \cdots, Z_p(a) \in T_0^\perp(\gamma(a))$, it follows that $Y_1(a) \in T_0(\gamma(a))$. This concludes the proof.

Added November 9, 1970. K. Nomizu and K. Abe independently obtained the completeness theorem above for curvature-like tensor fields by generalizing the methods of [1].

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