# A CHARACTERIZATION OF TWO-DIMENSIONAL RIEMANNIAN MANIFOLDS OF CONSTANT CURVATURE

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Let $\overline{M}$ be a Riemannian manifold, and let M be a *compact hypersurface*, that is, a compact orientable submanifold of codimension 1 of $\overline{M}$, possibly with boundary. (Everything is assumed to be $C^\infty$.) For sufficiently small s, let $M_s$ denote the set of points lying on geodesics normal to M (and on a fixed side of M) at distance s from M. Denoting the volume of $M_s$ by $\mathscr{A}(s)$, we call the real-valued function $\mathscr{A}$ (defined in a neighborhood of zero) the *growth function* of M. In [1], it is shown that $\mathscr{A}$ is a polynomial of degree at most 1, for each compact hypersurface in $\overline{M}$, if and only if $\overline{M}$ is locally isometric to $\mathbb{R}^2$. The purpose of the present note is to point out that the technique employed in [1] actually allows us to prove the following theorem, which is more general and more satisfactory.

THEOREM. *A Riemannian manifold has the property that the growth function* $\mathscr{A}$ *of each one of its compact hypersurfaces satisfies the linear differential equation*

$$\mathscr{A}'' + c\mathscr{A} = 0 \tag{1}$$

(*where* c *is a fixed constant*) *if and only if it is a two-dimensional Riemannian manifold of constant curvature equal to* c.

Using the known facts about the solutions of equation (1), we may rephrase the theorem in an equivalent way: the two-dimensional Riemannian manifolds of constant zero curvature are characterized by the fact that their growth functions are polynomials of degree at most 1; the two-dimensional Riemannian manifolds of constant positive curvature c are characterized by the fact that their growth functions are expressible as linear combinations of $\cos\sqrt{c}\,s$ and $\sin\sqrt{c}\,s$; and the two-dimensional Riemannian manifolds of constant negative curvature are characterized by the fact that their growth functions are expressible as linear combinations of $\cosh\sqrt{-c}\,s$ and $\sinh\sqrt{-c}\,s$.

Before giving the proof of the theorem, we must recall the results proved in [1]. We let M be a compact hypersurface of $\overline{M}$, and we let $M_s$ be as above. Denoting by $\Omega_s$ the volume form of M, we have by definition the relation

$$\mathscr{A}(s) = \int_{M_s} \Omega_s . \tag{2}$$

To state the formula for $\mathscr{A}''(s)$, we separate our discussion into two cases.

*Case* 1: $\dim \overline{M} = 2$. In this case, each $M_s$ is simply a finite $C^\infty$-curve. Let K denote the curvature function of the surface $\overline{M}$. Then

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$$\mathscr{A}''(s) = -\int_{M_s} K\Omega_s . \tag{3}$$

*Case* 2: $\dim \overline{M} \geq 3$. Let $\mathscr{R}$ denote the Ricci tensor of $\overline{M}$ ($\mathscr{R}$: $\overline{M}_m \to \overline{M}_m$), and let $h_s$: $(M_s)_m \otimes (M_s)_m \to \mathbb{R}$ denote the second fundamental form of $M_s$. Then $h_s$ admits an extension to

$$((M_s)_m \wedge (M_s)_m) \otimes ((M_s)_m \wedge (M_s)_m \to \mathbb{R} ,$$

which we also denote by $h_s$. Let $\dim \overline{M} = d$, and let $\{e_1, \cdots, e_{d-1}\}$ be some orthonormal basis of $(M_s)_m$. It is easy to see that $\sum_{ij} h_s(e_i \wedge e_j, e_i \wedge e_j)$ is a globally defined function on $M_s$, independent of the particular choice of $\{e_1, \cdots, e_{d-1}\}$. Finally, let $n_s$ denote the unit normal vector field to $M_s$ in the direction of increasing $s$. The second variation formula pertaining to this situation is

$$\mathscr{A}''(s) = \int_{M_s} \left( \sum_{ij} h_s(e_i \wedge e_j, e_i \wedge e_j) - \langle \mathscr{R}(n_s), n_s \rangle \right) \Omega_s . \tag{4}$$

Eventually, we shall also need the following lemma, proved at the end of [1].

LEMMA. *Sufficiently small geodesic spheres of any Riemannian manifold of dimension at least* 3 *have a positive definite second fundamental form.*

We can now give the simple proof of the theorem. Suppose $\overline{M}$ is a Riemannian manifold of dimension 2 whose curvature equals a constant $c$; then (2) and (3) imply that (1) holds. Conversely, suppose $\overline{M}$ has the property that its growth function $\mathscr{A}$ always satisfies (1). If $\overline{M}$ is of dimension 2, then (2) and (3) imply that for every finite curve $M$,

$$\int_M (K - c)\Omega = 0,$$

where we have denoted the volume form of $M$ by $\Omega$. Since this is true for *every* finite curve, it is obvious that $K = c$ and $\overline{M}$ has constant curvature $c$. It remains to show that if $\dim \overline{M} \geq 3$, the growth function $\mathscr{A}$ does not satisfy (1). If it does, then by (2) and (4),

$$\int_M \left\{ \sum_{ij} h(e_i \wedge e_j, e_i \wedge e_j) + c - \langle \mathscr{R}(n), n \rangle \right\} \Omega = 0$$

for every compact hypersurface $M$, where $h$ denotes the second fundamental form of $M$, $n$ is a unit normal field to $M$, and $\Omega$ is the volume form of $M$. As usual, the fact that this identity holds for *every* compact hypersurface $M$ simply means that

$$\sum_{ij} h(e_i \wedge e_j, e_i \wedge e_j) + c - \langle \mathscr{R}(n), n \rangle \equiv 0$$

on every hypersurface $M$ (compact or not). Now pick an arbitrary point $m$ of $\overline{M}$, and let $\{x_1, \cdots, x_d\}$ be a system of geodesic (normal) coordinates around $m$ satisfying the condition

$$\left\langle \frac{\partial}{\partial x_i}(m), \frac{\partial}{\partial x_j}(m) \right\rangle = \delta_{ij}.$$

Let M be the hypersurface defined by $x_d = 0$. It is well known that in this case the second fundamental form h of M at m vanishes. It follows that

$$h(e_i \wedge e_j, e_i \wedge e_j) = 0$$

for all $e_i, e_j \in M_m$, and (5) implies that

$$c - \left\langle \mathscr{R}\left(\frac{\partial}{\partial x_d}(m)\right), \frac{\partial}{\partial x_d}(m) \right\rangle = 0,$$

provided we choose the unit normal field n to coincide with $\frac{\partial}{\partial x_d}(m)$ at m. Now m is arbitrary, and $\frac{\partial}{\partial x_d}(m)$ can be any unit vector in $\overline{M}_m$; therefore $\overline{M}$ has constant Ricci curvature equal to c. Hence (5) implies that

$$\sum_{ij} h(e_i \wedge e_j, e_i \wedge e_j) \equiv 0$$

on every hypersurface M of $\overline{M}$. This contradicts the lemma quoted above, and the theorem is proved.

## REFERENCE

1. H. Wu, *A characteristic property of the euclidean plane.* Michigan Math. J. 16 (1969), 141-148.

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