QUANTITATIVE STABILITY OF DISCRETE SYSTEMS

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1. INTRODUCTION

The standard stability definitions for both continuous-time ([2], [4]) and discrete-time ([4], [8]) dynamical systems are qualitative in nature. Recently, several forms of stability of a more quantitative nature have been developed for continuous-time systems. These include finite time stability ([3], [5], [9], [15] to [17]), practical stability ([10], [11]), and set stability ([7], [12], [13]). In addition, for the discrete-time case, A. N. Michel and S. H. Wu [14] have introduced the concept of finite time stability and J. A. Heinen and Wu [6] the concept of set stability.

In this paper, we consider a very general form of quantitative stability (set stability) for discrete systems. In nonprecise terms, a discrete system is set stable if all solutions starting in a specific set remain for a specified time thereafter in a second specific (possibly time-varying) set. This form of stability thus has obvious connections with the estimation of transient response.

As is usually the case in stability theory, we shall approach the problem of set stability of discrete systems within the framework of the "Liapunov technique." Previously, R. E. Kalman and J. E. Bertram [8, Section 8], R. K. Cavin, C. L. Phillips, and D. L. Chenoweth [1], Michel and Wu [14], and Heinen and Wu [6] have applied this technique to obtain sufficient conditions for certain special cases of quantitative stability of discrete systems. In this paper, we develop necessary and sufficient conditions for quantitative stability in terms of the existence of suitable "discrete Liapunov functions." We also obtain necessary and sufficient conditions for instability of discrete systems.

2. SET STABILITY DEFINITIONS

We shall use the term *discrete system* to describe a first-order, n-dimensional vector difference equation

$$\text{(D)} \qquad \theta x = f(x, j),$$

where the operator θ, as in W. Hahn [4, p. 146], satisfies the condition $\theta x(j) = x(j+1)$ and where the integer j represents the independent (time) variable. In this equation, x belongs to R^n (n-dimensional Euclidean space) and f is a map $f: R^n \times J \to R^n$, where $J = [j_i, j_f]$ $(-\infty < j_i < j_f \leq +\infty)$ and where (as in the remainder of the paper) the following interval notation for sets of integers is employed:

$$[j_0, j_1] = \{j_0, j_0 + 1, \cdots, j_1\} \quad (-\infty < j_0 \leq j_1 < +\infty) \quad \text{and}$$

$$[j_0, \infty] = \{j_0, j_0 + 1, \cdots\} \quad (-\infty < j_0 < +\infty).$$

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A function $x: J \to R^n$ is called a *solution* of the discrete system (D) on J if, for each $j \in [j_i, j_f - 1]$, $x(j)$ satisfies the condition

$$x(j+1) = f(x(j), j).$$

The definitions of stability that follow are concerned with set-valued functions of j. In these definitions, S is of the form $S: J \to P(R^n)$, where $P(R^n)$ is the power set of R^n (that is, the collection of all subsets of R^n). Corresponding to each function S is its *complement function* $\tilde{S}: J \to P(R^n)$ defined, for all $j \in J$, by the relation $\tilde{S}(j) = \sim [S(j)] = \{x \in R^n: x \notin S(j)\}$.

Definition 1. A discrete system (D) is *set stable with respect to* $S: J \to P(R^n)$, $S_0 \in P(R^n)$, *and* J (abbreviated: $SS(S, S_0, J)$) if, whenever $x(j)$ is a solution of (D) on J, the relation $x(j_i) \in S_0$ implies $x(j) \in S(j)$, for all $j \in J$.

Definition 2. A discrete system (D) is *uniformly set stable with respect to* $S: J \to P(R^n)$, $S_0: J \to P(R^n)$, *and* J (abbreviated: $USS(S, S_0, J)$) if, whenever $j_0 \in J$ and $x(j)$ is a solution of (D) on $[j_0, j_f]$, the relation $x(j_0) \in S_0(j_0)$ implies $x(j) \in S(j)$, for all $j \in [j_0, j_f]$. Equivalently, (D) is $USS(S, S_0, J)$ if it is $SS(S, S_0(j_0), [j_0, j_f])$ for each $j_0 \in J$.

Remarks. (1) In the special case where

$$S(j) \equiv \{x: \|x\| < \beta\}, \qquad S_0 = \{x: \|x\| < \alpha\} \qquad (0 < \alpha \leq \beta),$$

and J is finite, the property $SS(S, S_0, J)$ is equivalent to the finite time stability with respect to $(\alpha, \beta, j_i, j_f - j_i, \|\cdot\|)$ of Michel and Wu [14].

(2) For each initial set S_0, the discrete system (D) is $SS(S, S_0, J)$ if and only if S is such that $x(j; S_0, j_i) \subset S(j)$ for all $j \in J$, where

$$x(j_1; S_0, j_0) = \{x(j_1): x(j) \text{ is a solution of (D) on } [j_0, j_f] \text{ with } x(j_0) \in S_0\}$$

$$(j_0 \leq j_1 \leq j_f).$$

Thus, for a prescribed S_0, the best quantitative stability result attainable is that (D) is $SS(x(\cdot; S_0, j_i), S_0, J)$, and this is, of course, always true. Similarly, for a prescribed $S_0(j)$, (D) is $USS(S, S_0, J)$ if and only if the inclusion

$$\bigcup_{j_0=j_i}^{j} x(j; S_0(j_0), j_0) \subset S(j)$$

holds for all $j \in J$.

(3) If (D) is $SS(S_k, S_{0_k}, J)$ for all k in some arbitrary index set K, then (D) is also $SS\left(\bigcup_{k\in K} S_k, \bigcup_{k\in K} S_{0_k}, J\right)$ and $SS\left(\bigcap_{k\in K} S_k, \bigcap_{k\in K} S_{0_k}, J\right)$, where

$$\left[\bigcup_{k\in K} S_k\right](j) = \bigcup_{k\in K} [S_k(j)] \qquad \text{and} \qquad \left[\bigcap_{k\in K} S_k\right](j) = \bigcap_{k\in K} [S_k(j)].$$

A similar statement can be made regarding uniform set stability.

3. STABILITY THEOREMS

The following theorems provide necessary and sufficient conditions for set stability in terms of the existence of discrete Liapunov functions, that is, functions of the form $V: R^n \times J \to R^1$. Corresponding to each discrete Liapunóv function V is its total difference evaluated along solutions of the discrete system (D) given by the equation

$$\Delta V_{(D)}(x, j) = V(f(x, j), j+1) - V(x, j).$$

For each solution x(j) of (D), the relationship

$$V(x(j+1), j+1) - V(x(j), j) = \Delta V_{(D)}(x(j), j)$$

holds for all $j \in [j_i, j_f - 1]$.

THEOREM 1. *Suppose* $S: J \to P(R^n)$ *and* $S_0 \subset S(j_i)$. *Then the discrete system* (D) *is* $SS(S, S_0, J)$ *if and only if there exists a function* $V: R^n \times J \to R^1$ *such that*

(a) $\Delta V_{(D)}(x, j) \leq 0$, *for all* $x \in S(j)$ *and all* $j \in [j_i, j_f - 1]$,

(b) $0 \leq V(x, j)$, *for all* $x \in \tilde{S}(j)$ *and all* $j \in [j_i + 1, j_f]$,

(c) $V(x, j_i) \leq 0$, *for all* $x \in S_0$, *and*

(d) *strict inequality holds in at least one of the three conditions* (a), (b), (c).

Proof. Sufficiency. Assume that a function V exists satisfying the conditions above. Let x(j) be some solution of (D) with $x(j_i) \in S_0$. Then, certainly, $x(j_i) \in S(j_i)$. Suppose that there exists a $j_1 \in [j_i + 1, j_f]$ such that $x(j_1) \notin S(j_1)$ (that is, $x(j_1) \in \tilde{S}(j_1)$) and such that $x(j) \in S(j)$ for all $j \in [j_i, j_1 - 1]$. Then, by hypotheses (b), (a), and (c), respectively, we obtain the inequalities

$$0 \leq V(x(j_1), j_1) \leq V(x(j_i), j_i) \leq 0.$$

By hypothesis (d), this becomes $0 < 0$, which is a contradiction. Hence no such j_1 exists, and $x(j) \in S(j)$ for all $j \in J$. Since x(j) is an arbitrary solution of (D) starting in S_0, (D) is $SS(S, S_0, J)$.

Necessity. Assume now that (D) is $SS(S, S_0, J)$. Define

$$V(x, j) = \begin{cases} 0 & \text{if } x \in x(j; S_0, j_i), \\ 1 & \text{otherwise}. \end{cases}$$

Suppose, first, that $x \in S(j)$ and $j \in [j_i, j_f - 1]$. Clearly, $\Delta V_{(D)}(x, j) \leq 0$, except in the case where $f(x, j) \notin x(j+1; S_0, j_i)$ and $x \in x(j; S_0, j_i)$. But this case cannot occur, by the definition of $x(j; S_0, j_i)$. Hence $\Delta V_{(D)}(x, j) \leq 0$, and condition (a) is satisfied. Now suppose $x \in \tilde{S}(j)$ and $j \in [j_i + 1, j_f]$. By Remark (2), since (D) is $SS(S, S_0, J)$, we have that $x(j; S_0, j_i) \subset S(j)$ for all $j \in J$. Hence $V(x, j) = 1 > 0$, and conditions (b) and (d) are satisfied. Finally, suppose $x \in S_0$. Then, since $S_0 = x(j_i; S_0, j_i)$, we have that $V(x, j_i) = 0$, and condition (c) is satisfied. This completes the proof.

An obvious consequence of Theorem 1, which often yields stronger quantitative stability results than Theorem 1, is the following corollary. This corollary can often be of use in connection with estimation of transient behavior.

COROLLARY 1. *If a function* V *exists satisfying the conditions of Theorem* 1, *then* (D) *is not only* SS(S, S_0, J) *but also* SS(S_v, S_{v_0}, J), *where*

$$S_v(j) = \{x \in S(j): V(x, j) \leq 0\} \quad \text{and} \quad S_{v_0} = \{x \in S(j_i): V(x, j_i) < 0\} \cup S_0 .$$

THEOREM 2. *Suppose the functions* S, S_0: J $\to$ P(R^n) *are such that* $S_0(j) \subset S(j)$ *for all* $j \in J$. *Then the discrete system* (D) *is* USS(S, S_0, J) *if and only if there exists a function* V: $R^n \times J \to R^1$ *such that*

(a) $\Delta V_{(D)}(x, j) \leq 0$, *for all* $x \in S(j)$ *and all* $j \in [j_i, j_f - 1]$,

(b) $0 \leq V(x, j)$, *for all* $x \in \tilde{S}(j)$ *and all* $j \in [j_i + 1, j_f]$,

(c) $V(x, j) \leq 0$, *for all* $x \in S_0(j)$ *and all* $j \in [j_i, j_f - 1]$, *and*

(d) *strict inequality holds in at least one of the three conditions* (a), (b), (c).

Proof. Sufficiency. If a function V exists satisfying the conditions above, then the conditions of Theorem 1 are clearly met on every interval $[j_0, j_f] \subset J$. Hence (D) is SS(S, $S_0(j_0)$, $[j_0, j_f]$) for each $j_0 \in J$. But this is to say (D) is USS(S, S_0, J).

Necessity. This part of the proof is similar to the corresponding proof of Theorem 1, with the exception that now we choose V to be

$$V(x, j) = \begin{cases} 0 & \text{if } x \in \bigcup_{j_0=j_i}^{j} x(j; S_0(j_0), j_0), \\ 1 & \text{otherwise}. \end{cases}$$

COROLLARY 2. *If a function* V *exists satisfying the conditions of Theorem* 2, *then* (D) *is not only* USS(S, S_0, J) *but also* USS(S_v, S_{v_0}, J), *where*

$$S_v(j) = \{x \in S(j): V(x, j) \leq 0\} \cup S_0(j) \quad \text{and} \quad S_{v_0}(j) = \{x \in S(j): V(x, j) < 0\} \cup S_0(j).$$

Note that in the theorems above no assumptions have been made regarding the continuity of V, the continuity of f, the uniqueness of solutions of (D), the existence of equilibria of (D), or the classical stability of (D). Furthermore, J can be either finite or infinite, and it is entirely possible for the sets S and S_0 to be unbounded.

4. INSTABILITY THEOREMS

The theorems of this section provide necessary and sufficient conditions under which a fixed discrete system is not set stable. The proofs of these instability theorems are omitted, since they follow readily from Theorem 1 and the definitions of stability.

THEOREM 3. *Suppose* S: J $\to$ P(R^n) *and* $S_0 \subset S(j_i)$. *Then the discrete system* (D) *is not* SS(S, S_0, J) *if and only if there exist a function* Q: J $\to$ P(R^n), *an integer* $j_1 \in [j_i + 1, j_f]$, *and a function* V: $R^n \times J \to R^1$ *such that* $Q(j_i) \cap S_0 \neq \emptyset$ (the null set), $Q(j_1) \cap S(j_1) = \emptyset$, *and such that*

(a) $\Delta V_{(D)}(x, j) \leq 0$, *for all* $x \in Q(j)$ *and all* $j \in [j_i, j_1 - 1]$,

(b) $0 \leq V(x, j)$, *for all* $x \in \tilde{Q}(j)$ *and all* $j \in [j_i + 1, j_1]$,

(c) $V(x_0, j_i) \leq 0$, *for at least one* $x_0 \in Q(j_i) \cap S_0$, *and*

(d) *strict inequality holds in at least one of the three conditions* (a), (b), (c).

THEOREM 4. *Suppose the functions* $S, S_0: J \to P(R^n)$ *are such that* $S_0(j) \subset S(j)$ *for all* $j \in J$. *Then the discrete system* (D) *is not* USS(S, S_0, J) *if and only if there exist a function* $Q: J \to P(R^n)$, *integers* j_0 *and* j_1 *in* J ($j_0 < j_1$), *and a function* $V: R^n \times J \to R^1$ *such that* $Q(j_0) \cap S_0(j_0) \neq \emptyset$, $Q(j_1) \cap S(j_1) = \emptyset$, *and such that*

(a) $\Delta V_{(D)}(x, j) \leq 0$, *for all* $x \in Q(j)$ *and all* $j \in [j_0, j_1 - 1]$,

(b) $0 \leq V(x, j)$, *for all* $x \in \tilde{Q}(j)$ *and all* $j \in [j_0 + 1, j_1]$,

(c) $V(x_0, j_0) \leq 0$, *for at least one* $x_0 \in Q(j_0) \cap S_0(j_0)$, *and*

(d) *strict inequality holds in at least one of the three conditions* (a), (b), (c).

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