AN OSCILLATION CRITERION FOR SOLUTIONS OF $(ry')' + qy^{\gamma} = 0$

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For the differential equation

$$(1) \qquad y'' + fy^{2n-1} = 0$$

with $f > 0$ and $n = 2, 3, \cdots$, F. V. Atkinson [1] proved that all nontrivial solutions are oscillatory if and only if $\int^{\infty} t f(t)\,dt = \infty$. In the special case $f(t) = t^{\beta}$, R. H. Fowler [3] proved that (1) has an oscillatory solution if and only if $\beta \geq -(n+1)$. With this f, the criterion of Atkinson is equivalent to the condition $\beta \geq -2$; thus (1) may have both oscillatory and nonoscillatory solutions. A result of J. Kurzweil [5] applies to (1) when nonoscillatory solutions exist; it states that (1) has an oscillatory solution if $f(t)t^{n+1}$ is nondecreasing. In this note we give an oscillation criterion for a generalization of (1); it avoids the monotonicity hypothesis on the coefficients.

We consider the differential equation

$$(2) \qquad (ry')' + qy^{\gamma} = 0,$$

where $\gamma \geq 1$ is the ratio of odd, positive integers. It is assumed throughout that r and q are positive on a ray $[a, \infty)$, and that they have two continuous derivatives. Thus a local existence and uniqueness theorem holds; moreover, all solutions of (2) are extendable to $[a, \infty)$ (see [4]). Hence, for each choice of initial values y(a) and y'(a), we have a unique solution of (2) on $[a, \infty)$.

THEOREM. *If* (i) $K = \int_a^{\infty} |\eta(r\eta')'|\,dt < \infty$ *and* (ii) $\int_a^{\infty} \frac{1}{r\eta^2}\,dt = \infty$, *where* $\eta(t) = [r(t)q(t)]^{-1/(\gamma+3)}$, *then equation* (2) *has an oscillatory solution. Moreover, for* $\gamma > 1$, *every solution* y *of* (2) *satisfying the inequality*

$$|y(a)| > \eta(a)[2(\gamma+1)K^2]^{1/(\gamma-1)}$$

is oscillatory.

Proof. Let $h(t) = \int_a^t \frac{1}{r\eta^2}\,dt$, and let y be a solution of (2) such that $|y(a)| > \eta(a)[2(\gamma+1)K^2]^{1/(\gamma-1)}$ if $\gamma > 1$, and such that $y(a) \neq 0$ if $\gamma = 1$. Define the function x implicitly on $[0, \infty)$ by the equation $y(t) = \eta(t)x(h(t))$. A calculation and the relation $h' = \frac{1}{r\eta^2}$ show that

$$(3) \qquad 0 = (ry')' + qy^{\gamma}, \qquad 0 = (x'' \circ h) + (r\eta')'(r\eta^3)(x \circ h) + (x \circ h)^{\gamma}.$$

Let $z(s) = (1/2)[x'(s)]^2 + [x(s)]^{\gamma+1}/(\gamma+1)$ for $s \geq 0$; then $z(s) \geq 0$ for all $s \geq 0$. We now prove that z is bounded on $[0, \infty)$. The proof proceeds as that of Theorem 1

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of [4]. If z is unbounded, then there exists an increasing sequence $s_1, s_2, \cdots$ such that $z(s_i) > 1$, $z(s_i) \to \infty$ as $i \to \infty$, and

$$z(s_i) = \max\{z(s): 0 \le s \le s_i\} \qquad (i = 1, 2, \cdots).$$

For $s = h(t)$ it follows from (3) that

$$z'(s) = x'(s)x''(s) + x'(s)x(s)^\gamma = -[r\eta^3(r\eta')'](h^{-1}(s))x(s)x'(s). \tag{4}$$

Since $|x'(s)| \le [2z(s)]^{1/2}$, $|x(s)| \le [(\gamma+1)z(s)]^{1/(\gamma+1)}$, and $z(s_i) > 1$, it follows from (4) that for $a < b$ and $h(b) < s_i$,

$$\begin{aligned} z(s_i) &\le z(h(b)) + \int_{h(b)}^{s_i} |[r\eta^3(r\eta')'](h^{-1}(s))x'(s)x(s)|\,ds \\ &\le z(h(b)) + \sqrt{2}(\gamma+1)^{1/(\gamma+1)} \cdot \left[\int_b^{h^{-1}(s_i)} |\eta(r\eta')'|\,dt\right] \cdot z(s_i)^{(\gamma+3)/2(\gamma+1)} \\ &\le z(h(b)) + \sqrt{2}(\gamma+1)^{1/(\gamma+1)} \left[\int_b^\infty |\eta(r\eta')'|\,dt\right] z(s_i). \end{aligned}$$

Choosing b so that

$$\sqrt{2}(\gamma+1)^{1/(\gamma+1)} \left[\int_b^\infty |\eta(r\eta')'|\,dt\right] < 1/2,$$

we obtain a contradiction to $z(s_i) \to \infty$ as $i \to \infty$. It follows from the boundedness of z and integrability of $\eta(r\eta')'$ that $z(s) \to L$ as $s \to \infty$, for some number $L \ge 0$.

We now prove that $L > 0$. First consider $\gamma = 1$; then equation (4) is of the form $z'(s) = A(s)z(s)$, where

$$A(s) = -[r\eta^3(r\eta')'](h^{-1}(s))x(s)x'(s)/z(s).$$

From the definition of z we see that $|x(s)x'(s)| \le 2z(s)$; thus

$$\int_0^\infty |A(s)|\,ds \le 2\int_a^\infty |\eta(r\eta')'|\,dt < \infty.$$

This implies that $z(s) \to z(0)\exp\left[\int_0^\infty A(s)\,ds\right] \ne 0$, as $s \to \infty$. Suppose now that $\gamma > 1$ and $L = 0$. Let

$$L_1 = \text{l.u.b.}\{|x'(s)|: 0 \le s < \infty\} \quad \text{and} \quad L_2 = \text{l.u.b.}\{|x(s)|: 0 \le s < \infty\}.$$

From (4) and the condition $L = 0$ we deduce that

$$z(s) = \int_s^\infty [r\eta^3(r\eta')'](h^{-1}(v))x'(v)x(v)\,dv \le KL_1L_2.$$

This inequality and the definition of z imply that $L_1^2/2 \le KL_1L_2$ and $L_2^{\gamma+1}/(\gamma+1) \le KL_1L_2$; hence we conclude that $L_2^{\gamma-1} \le 2(\gamma+1)K^2$. This inequality yields the inequality

$$|y(a)/\eta(a)| = |x(0)| \le L_2 \le [2(\gamma+1)K^2]^{1/(\gamma-1)},$$

but we chose the solution y of (2) so that $|y(a)| > \eta(a)[2(\gamma+1)K^2]^{1/(\gamma-1)}$, and the contradiction shows that $L > 0$.

Finally, we prove that $L > 0$ implies that x and hence y are oscillatory. Choose S so that $z(s) \ge L/2$ for $s \ge S$, and suppose x has no zeros on $[S, \infty)$. If x' has infinitely many zeros $u_1 < u_2 < \cdots$ on $[S, \infty)$, then the minimum of $|x|$ on $[u_1, u_i]$ occurs at a zero of x', and it is therefore not less than $[(\gamma+1)L/2]^{1/(\gamma+1)}$. Hence $|x(s)| \ge [(\gamma+1)L/2]^{1/(\gamma+1)}$ on $[u_1, \infty)$. The equation

$$x'(u) = \int_{u_1}^{u} x''(s)\,ds = -\int_{u_1}^{u} \{[r\eta^3(r\eta')'](h^{-1}(s))x(s) + x(s)^\gamma\}\,ds$$

shows that $|x'(u)| \to \infty$ as $u \to \infty$, a contradiction. Thus x is eventually monotone, and $x(s) \to M$ as $s \to \infty$. The above argument implies that $M = 0$. Hence $|x'(s)| \to (2L)^{1/2}$ as $s \to \infty$, which yields the contradiction that x is unbounded. Therefore x is oscillatory.

We note that in the proof above the only use of the initial value of y(a) was in proving $L > 0$. Hence every solution y of (2) for which the corresponding function x yields a number $L > 0$ is oscillatory.

COROLLARY. *If* $r \equiv 1$, $\sigma = (\gamma+5)/(\gamma+3)$, *and* $\int_a^\infty |q^{-\sigma} q''|\,dt < \infty$, *then* (2) *has an oscillatory solution.*

Proof. Let $p = q^{\sigma+1}$ and $\mu = q^{-\sigma}$. Then $\mu(p\mu')' = -\sigma q^{-\sigma} q''$, and the hypothesis of Lemma 5 on page 119 of [2] is satisfied. We have the equation

$$p(\mu')^2 = \sigma^2 q^{-(\sigma+1)}(q')^2,$$

and if $\int_a^\infty p(\mu')^2\,dt = \infty$, then by Lemma 5,

$$-\sigma q'(t) = (p\mu')(t) \to \gamma^* \quad \text{as } t \to \infty;$$

therefore $\gamma^* > 0$, since $\mu > 0$. This, however, implies that q is eventually negative. Thus $\int_a^\infty p(\mu')^2\,dt < \infty$, and since

$$\eta\eta'' = (\gamma+4)(\gamma+3)^{-2} q^{-(\sigma+1)}(q')^2 - (\gamma+3)^{-1} q^{-\sigma} q'',$$

condition (i) above is satisfied. Again by Lemma 5,

$$(p\mu\mu')(t) = -\sigma\,[q(t)]^{-\sigma} q'(t) \to \delta \quad \text{as } t \to \infty.$$

Thus by L'Hospital's rule,

$$\frac{1}{t\,q(t)^{\sigma-1}} = \frac{q(t)^{-(\sigma-1)}}{t} \to \frac{(\sigma - 1)\delta}{\sigma} \quad \text{as } t \to \infty;$$

from this we conclude that $\int_a^\infty q(t)^{\sigma-1}\,dt = \int_a^\infty \eta(t)^{-2}\,dt = \infty$, and condition (ii) above is satisfied.

For $q(t) = t^\beta$, the condition $\int_a^\infty |q^{-\sigma} q''|\,dt < \infty$ is equivalent to the condition $\beta > -(\gamma + 3)/2 = -(n + 1)$, when $\gamma = 2n - 1$. Choosing $q(t) = t^{-3}[1 + (1/2)\sin t^{1/4}]$ and $\gamma = 11$, we obtain an example where $\int_a^\infty |q^{-\sigma} q''|\,dt < \infty$ and $q(t)t^{n+1} = q(t)t^7$ is not nondecreasing.

REFERENCES

1. F. V. Atkinson, *On second-order non-linear oscillations.* Pacific J. Math. 5 (1955), 643-647.

2. W. A. Coppel, *Stability and asymptotic behavior of differential equations.* D. C. Heath, Boston, 1965.

3. R. H. Fowler, *Further studies of Emden's and similar differential equations.* Quart. J. Math. 2 (1931), 259-288.

4. D. B. Hinton, *Some stability conditions for a nonlinear differential equation.* Trans. Amer. Math. Soc. 139 (1969), 349-358.

5. J. Kurzweil, *A note on oscillatory solutions of the equation* $y'' + f(x)y^{2n-1} = 0$. Časopis Pěst. Mat. 85 (1960), 357-358.

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