SOME TOPOLOGICAL INVARIANTS OF STONE SPACES

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Most of the work of this paper was motivated by a still unsolved problem in homological algebra: Let J be an ideal in the commutative regular ring R, and let U[J] be the corresponding open subset of the maximal ideal space X(R). Is the projective dimension of J a topological invariant of U[J]? It can be shown that J is projective if and only if U[J] satisfies the equivalent conditions of Theorem 2.2. More generally, the projective dimension of J is at most equal to the cohomological dimension of U[J], which in turn is at most equal to the covering dimension of U[J]. In Sections 2 and 5, we show by examples that the covering dimension of U[J] may be strictly greater than the cohomological dimension of U[J]; but no example is known to the author in which the cohomological dimension of U[J] is strictly greater than the projective dimension of J. Most of the paper is concerned with a purely topological investigation of the covering dimension and cohomological dimension of Stone spaces, that is, of open subsets of Boolean spaces. (Stone spaces may be characterized equivalently as locally compact, totally disconnected Hausdorff spaces.)

1. COVERING DIMENSION

Let $\mathcal{U}$ be an open cover of the topological space X. The *order* $o(\mathcal{U})$ is the largest integer n such that there exist $n + 1$ distinct members of $\mathcal{U}$ with nonempty intersection. If no such integer exists, we say $o(\mathcal{U}) = \infty$. The *covering dimension* cov dim X is the least integer n such that every open cover of X has an open refinement of order at most n; if no such integer exists, we set cov dim $X = \infty$. Suppose X is a Boolean space, that is, a compact, totally disconnected Hausdorff space. It is well known (and easily proved) that each open cover of X has a finite, disjoint refinement consisting of compact, open sets. We have the following analogue for Stone spaces:

LEMMA 1.1. *Let* X *be a Stone space, and assume* X *has a compact, open cover of order* n. *Then each open cover of* X *has a compact, open refinement of order at most* n. *In particular,* cov dim $X \leq n$.

Proof. Let $\mathcal{U}$ be an open cover of X. Choose a compact, open cover $\mathcal{V}$ of order n, and for each $V \in \mathcal{V}$, let $\mathcal{W}_V$ be a (finite) disjoint, compact, open cover of V that refines $\mathcal{U}$. Then $\mathcal{W} = \bigcup \{\mathcal{W}_V: V \in \mathcal{V}\}$ is a compact, open refinement of $\mathcal{U}$, and $o(\mathcal{W}) \leq n$.

THEOREM 1.2. *Let* X *be a Stone space. Then* cov dim $X \leq n$ *if and only if* X *has a compact, open cover of order at most* n.

Proof. One implication follows from Lemma 1.1. To prove the converse, let $\mathcal{U}$ be a compact, open cover of X, and let $\mathcal{V} = \{V_i: i \in I\}$ be an open refinement of order at most n. Since each V_i has compact closure and $\mathcal{V}$ is point-finite, an easy argument (similar to [3, Problem 5.V]) based on Zorn's lemma shows that $\mathcal{V}$ can be shrunk to a compact, open cover $\mathcal{W} = \{W_i: i \in I\}$, with $W_i \subseteq V_i$ for each $i \in I$. Clearly, $o(\mathcal{W}) \leq n$.

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Recall that a space is *metacompact* provided every open cover has a point-finite open refinement. By the same reasoning as above, we obtain the following theorem.

THEOREM 1.3. *A Stone space is metacompact if and only if it has a point-finite, compact, open cover.*

THEOREM 1.4. *A Stone space is paracompact if and only if its covering dimension is* 0.

Proof. Every space of covering dimension 0 is obviously paracompact. Conversely, assume X is a paracompact Stone space, let $\mathscr{U}$ be a compact, open cover of X, and let $\mathscr{V} = \{V_i: i \in I\}$ be a locally finite refinement of $\mathscr{U}$. As in the proof of Theorem 1.2, we can shrink $\mathscr{V}$ to a compact, open cover $\mathscr{W} = \{W_i: i \in I\}$. For each $x \in X$, let N_x be a neighborhood of x that intersects only finitely many of the sets W_i. For each i, let $\mathscr{B}_i$ be a finite, disjoint, compact, open cover of W_i that refines $\{N_x: x \in X\}$, and let $\mathscr{B} = \bigcup_{i \in I} \mathscr{B}_i$. Well-order $\mathscr{B}$, say $\mathscr{B} = \{B_\xi: \xi < \lambda\}$. Now each B_ξ, since it is contained in some N_x, intersects only finitely many W_i and therefore only finitely many B_η. It follows that $C_\xi = B_\xi \setminus \bigcup_{\eta < \xi} B_\eta$ is a compact, open subset of X. Thus $\mathscr{C} = \{C_\xi: \xi < \lambda\}$ is a disjoint, compact, open cover of X.

2. COHOMOLOGICAL DIMENSION

Let $\mathscr{A}$ be a sheaf of abelian groups over the space X. Let $H^*(X; \mathscr{A})$ denote the natural cohomology of X with coefficients in $\mathscr{A}$, as defined in [8]. Let $H^*(\mathscr{U}; \mathscr{A})$ and $\check{H}^*(X; \mathscr{A})$, respectively, denote the cohomology of the open cover $\mathscr{U}$ and the Čech cohomology of X, with coefficients in $\mathscr{A}$. We record the following result, which is Theorem 4.1 of [10].

THEOREM 2.1. *Let* $\mathscr{A}$ *be a sheaf of abelian groups over the Stone space* X, *and let* $\mathscr{U}$ *be a compact, open cover of* X. *Then, for each* $n \geq 0$, *the natural homomorphisms*

$$H^n(\mathscr{U}; \mathscr{A}) \to \check{H}^n(X; \mathscr{A}) \to H^n(X; \mathscr{A})$$

are isomorphisms.

Let dim X denote the *cohomological* dimension of X, that is, the largest integer n such that $H^n(X; \mathscr{A}) \neq 0$ for some sheaf of abelian groups $\mathscr{A}$ over X. If no such integer exists, we set $\dim X = \infty$.

Combining Theorem 1.4 with [10, Theorem 5.1], we obtain the following result.

THEOREM 2.2. *The following three conditions on a Stone space* X *are equivalent:*

(a) $\dim X = 0$, (b) $\text{cov dim } X = 0$, (c) X *is paracompact.*

By [10, Example 5.4], there exist Stone spaces of infinite cohomological dimension. Our goal in this section is to exhibit, for each $n \geq 0$, a Stone space with cohomological dimension n. We need an upper bound on dim X. (Of course, $\text{cov dim } X \geq \dim X$, but the spaces we shall consider have infinite covering dimension.)

Definition. The *rank* of a topological space X is the least integer n such that X can be expressed as the union of $\aleph_n$ compact subsets. If no such integer exists, rank $X = \infty$.

The following result was announced in [9]. We shall give the proof in the next section.

THEOREM 2.3. *Let* X *be a Stone space. Then* $\dim X \leq \operatorname{rank} X$.

Example 2.4. Let n be a nonnegative integer, and let Y_n be the Cantor space $2^{\aleph_n}$. Let X_n be the open subset of Y_n obtained by deletion of the point all of whose coordinates are 1. The Boolean ring of Y_n is the free Boolean ring on $\aleph_n$ generators, and the ideal corresponding to X_n has homological dimension n, by [7, Theorem 5.1]. It follows immediately from [10, Theorem 2.1] that $\dim X_n \geq n$. But X_n is covered by the $\aleph_n$ compact sets $\pi_\alpha^{-1}(0)$, where $\pi_\alpha\colon \aleph_n \to \{0, 1\}$ is the α^{th} projection map. Therefore, by Theorem 2.3, the cohomological dimension of X_n is precisely n.

We remarked earlier that the spaces X_n $(n \geq 1)$ have infinite covering dimension. One way to see this is as follows: Let X be a Stone space with a compact open basis of cardinality at most $\aleph_n$, and let X^* be its one-point compactification. Then one can clearly define a one-to-one continuous map $\phi\colon X^* \to Y_n$ in such a way that $\phi(X) = \phi(X^*) \cap X_n$, that is, X is homeomorphic to a closed subset of X_n. In particular, let X be the space of countable ordinals with the order topology. Then X is not metacompact, since X is countably compact but not compact [3, Problems 5.E, 5.V]. It follows that X_n $(n \geq 1)$ is not metacompact.

3. PROOF OF THEOREM 2.3

Let X be a topological space. By a *well-ordered open cover* of X we shall mean an open cover $\{U_\alpha\colon \alpha < \lambda\}$, indexed by an ordinal λ and satisfying the conditions i) $U_\alpha \subseteq U_\beta$ if $\alpha \leq \beta < \lambda$ and ii) $U_\gamma = \bigcup_{\alpha<\gamma} U_\alpha$ if γ is a limit ordinal.

PROPOSITION 3.1. *Let* $\mathscr{A}$ *be a sheaf of abelian groups over* X. *Suppose* X *has a well-ordered open cover* $\{U_\alpha\colon \alpha < \lambda\}$ *such that*

$$H^n(U_\alpha; \mathscr{A}) = H^{n+1}(U_\alpha; \mathscr{A}) = 0 \quad \textit{for each } \alpha < \lambda.$$

Then $H^{n+1}(X; \mathscr{A}) = 0$.

Proof. Let $\mathscr{F}^*\colon 0 \to \mathscr{A} \to \mathscr{F}^0 \to \mathscr{F}^1 \to \cdots$ be a flabby resolution of $\mathscr{A}$. For each $\alpha < \lambda$, let $\mathscr{A}_\alpha = \mathscr{A} \mid U_\alpha$, and let $\mathscr{F}^i_\alpha = \mathscr{F}^i \mid U_\alpha$. The induced sequence $\mathscr{F}^*_\alpha\colon 0 \to \mathscr{A}_\alpha \to \mathscr{F}^0_\alpha \to \mathscr{F}^1_\alpha \to \cdots$ is then a flabby resolution of $\mathscr{A}_\alpha$. Therefore $H^p(U_\alpha; \mathscr{A})$ is isomorphic to the p^{th} cohomology group of the complex

$$\Gamma\mathscr{F}^*\colon 0 \to \Gamma(U_\alpha, \mathscr{F}^0) \to \Gamma(U_\alpha, \mathscr{F}^1) \to \cdots.$$

In particular, the sequence

$$\Gamma(U_\alpha, \mathscr{F}^{n-1}) \xrightarrow{\delta} \Gamma(U_\alpha, \mathscr{F}^n) \xrightarrow{\delta} \Gamma(U_\alpha, \mathscr{F}^{n+1}) \xrightarrow{\delta} \Gamma(U_\alpha, \mathscr{F}^{n+2}) \tag{1}$$

is exact for each $\alpha < \lambda$. (In order to treat the case $n = 0$, let $\mathscr{F}^{-1} = 0$.) We must verify that the sequence

$$\Gamma(X, \mathscr{F}^n) \xrightarrow{\delta} \Gamma(X, \mathscr{F}^{n+1}) \xrightarrow{\delta} \Gamma(X, \mathscr{F}^{n+2}) \tag{2}$$

is exact. Let $s \in \Gamma(X, \mathscr{F}^{n+1})$, and suppose $\delta s = 0$. For each $\alpha < \lambda$, let $s_\alpha = s \mid U_\alpha$. We shall define, inductively, sections $t_\alpha \in \Gamma(U_\alpha, \mathscr{F}^n)$ such that $\delta t_\alpha = s_\alpha$ and $t_\beta = t_\alpha \mid U_\beta$ for $\beta < \alpha$. This will show that the sequence (2) is exact, and the proof will be complete.

Let t_0 be an element of $\Gamma(U_0, \mathscr{F}^n)$ whose coboundary is s_0. Suppose t_α has been chosen for each $\alpha < \gamma$. If γ is a limit ordinal, condition ii) and the compatibility of the t_α provide an obvious choice for a satisfactory t_γ. On the other hand, if $\gamma = \beta + 1$, choose an $r_\gamma \in \Gamma(U_\gamma, \mathscr{F}^n)$ such that $\delta r_\gamma = s_\gamma$. Then $\delta(r_\gamma \mid U_\beta - t_\beta) = 0$; hence there is a section $p_\beta \in \Gamma(U_\beta, \mathscr{F}^{n-1})$ such that $\delta p_\beta = r_\gamma \mid U_\beta - t_\beta$. Since $\mathscr{F}^{n-1}$ is flabby, p_β can be extended to $p_\gamma \in \Gamma(U_\gamma, \mathscr{F}^{n-1})$. Now let $t_\gamma = r_\gamma - \delta p_\gamma$. Clearly, $\delta t_\gamma = s_\gamma$, and $t_\gamma \mid U_\alpha = t_\alpha$ for each $\alpha < \gamma$.

COROLLARY 3.2. *Let* X *be a topological space that has a well-ordered open cover consisting of sets with cohomological dimension at most* n. *Then* $\dim X \leq n + 1$.

We can now give a simple proof of Theorem 2.3. (Our method was suggested by the proof of a corresponding theorem on the global dimension of regular rings [7], which is based on a module-theoretic analogue of Corollary 3.2 due to M. Auslander [1].) Since every compact subset of a Stone space X is contained in a compact open set, we see that $\operatorname{rank} X \leq n$ if and only if X has a compact open cover of cardinality at most $\aleph_n$. If $\operatorname{rank} X = 0$, clearly $\operatorname{cov} \dim X = 0$; therefore $\dim X = 0$. Since every Stone space of rank n has a well-ordered open cover consisting of open sets of rank at most n - 1, the theorem follows by induction on n.

4. SPACES WITH FINITE COVERING DIMENSION

In this section we show that there exist Stone spaces of arbitrary covering dimension. The following theorem says somewhat more.

THEOREM 4.1. *For each* $n \geq 0$, *there exists a Stone space with rank* n *and covering dimension* n *that is the union of* $n + 1$ *paracompact open sets.*

Proof. Let $A_0, \cdots, A_n$ be pairwise disjoint, well-ordered sets such that A_k is order-isomorphic to $\aleph_k$ $(0 \leq k \leq n)$. Give each A_k the discrete topology, and let $B_k = A_k \cup \{*\}$ be the one-point compactification of A_k. Let Y_n be the cartesian product $B_0 \times \cdots \times B_n$, and let $X_n = Y_n \setminus \{(*, \cdots, *)\}$. For each $i \leq n$ and each $\xi \in A_i$, let

$$U(i, \xi) = B_0 \times \cdots \times B_{i-1} \times \{\xi\} \times B_{i+1} \times \cdots \times B_n,$$

and let $U_i = \bigcup \{U(i, \xi) \mid \xi \in A_i\}$. Then $X_n = U_0 \cup \cdots \cup U_n$, and since each U_i is a disjoint union of compact open sets, the last assertion follows. Let

$$\mathscr{U} = \{U(i, \xi) \mid 0 \leq i \leq n,\ \xi \in A_i\}.$$

Then $\mathscr{U}$ is a compact open cover of X_n of order n. Suppose $\mathscr{V}$ is an open cover of X_n that refines $\mathscr{U}$. We shall show that $o(\mathscr{V}) \geq n$.

For each pair (i, ξ) with $\xi \in A_i$, choose an open set $V(i, \xi) \in \mathscr{V}$ that contains the point $x(i, \xi) = (*, \cdots, *, \xi, *, \cdots, *)$. Since $U(i, \xi)$ is the only member of $\mathscr{U}$ that contains $x(i, \xi)$, it follows that $V(i, \xi) \subseteq U(i, \xi)$ and that the $V(i, \xi)$ are all distinct. We shall show that there exists a point $(\xi_0, \cdots, \xi_n) \in A_0 \times \cdots \times A_n$ that lies in $V(0, \xi_0) \cap \cdots \cap V(n, \xi_n)$.

By construction, $(\xi_0, *, \cdots, *) \in V(0, \xi_0)$, for each $\xi_0 \in A_0$. Assume inductively that there exist $\sigma_k \in A_k$ and functions $\sigma_i: A_{i+1} \times \cdots \times A_k \to A_i$ $(0 \leq i \leq k-1)$ such that $(\xi_0, \cdots, \xi_k, *, \cdots, *) \in V(0, \xi_0) \cap \cdots \cap V(k, \xi_k)$ whenever

$$\text{(3)} \qquad \xi_k > \sigma_k, \quad \xi_{k-1} > \sigma_{k-1}(\xi_k), \quad \cdots, \quad \xi_0 > \sigma_0(\xi_1, \cdots, \xi_k).$$

If $k = n$, we have finished; therefore assume that $k < n$. Since only $\aleph_k$ sequences $(\xi_0, \cdots, \xi_k)$ satisfy (3), there must exist some $\tau_{k+1} \in A_{k+1}$ so large that

$$(\xi_0, \cdots, \xi_k, \xi_{k+1}, *, \cdots, *) \in V(0, \xi_0) \cap \cdots \cap V(k, \xi_k)$$

whenever $\xi_{k+1} > \tau_{k+1}$ and $(\xi_0, \cdots, \xi_k)$ satisfies (3). Also, since

$$(*, \cdots, *, \xi_{k+1}, *, \cdots, *) \in V(k+1, \xi_{k+1})$$

for each $\xi_{k+1} \in A_{k+1}$, there exists $\rho_i(\xi_{k+1}) \in A_i$, for each $i \leq k$, such that

$$(\xi_0, \cdots, \xi_k, \xi_{k+1}, *, \cdots, *) \in V(k+1, \xi_{k+1}) \quad \text{whenever } \xi_i > \rho_i(\xi_{k+1}) \ (0 \leq i \leq k).$$

Now let $\tau_k(\xi_{k+1}) = \max\{\sigma_k, \rho_k(\xi_{k+1})\}$, and for $i < k$, let

$$\tau_i(\xi_{i+1}, \cdots, \xi_{k+1}) = \max\{\sigma_i(\xi_{i+1}, \cdots, \xi_k), \rho_i(\xi_{k+1})\}.$$

Then, if $\xi_{k+1} > \tau_{k+1}$ and $\xi_i > \tau_i(\xi_{i+1}, \cdots, \xi_{k+1})$ for $0 \leq i \leq k$, we have the relation

$$(\xi_0, \cdots, \xi_{k+1}, *, \cdots, *) \in V(0, \xi_0) \cap \cdots \cap V(k+1, \xi_{k+1}).$$

This completes the proof that cov dim $X_n = n$. We need only verify that rank $X_n = n$. Clearly, rank $X \leq n$. The opposite inequality follows from the observation that each compact open set contains at most finitely many of the $\aleph_n$ points $x(n, \xi)$ $(\xi \in A_n)$.

5. DECOMPOSITION OF STONE SPACES

The preceding example raises two questions:

1) If cov dim $X = n < \infty$, is the rank of X necessarily at least n?

2) Is every Stone space of covering dimension n the union of $n+1$ paracompact open subsets?

In this section we answer both questions negatively. We first record the following observation, which is an immediate consequence of Theorem 1.2.

PROPOSITION 5.1. *Let* U *and* V *be open subsets of a Stone space, with covering dimensions* r *and* s, *respectively. Then* cov dim$(U \cup V) \leq r + s + 1$.

Definition. Let X be a Stone space of covering dimension n. We say that X is *completely decomposable* if X is the union of $n+1$ paracompact open subsets; X is *indecomposable* if X is not the union of two open sets of covering dimension less than n.

If X is a completely decomposable Stone space of covering dimension n (say, the space X_n of Theorem 4.1), and if r and s are nonnegative integers whose sum is $n-1$, then, by Proposition 5.1, we can write $X = U \cup V$, where U and V are open subsets with covering dimensions r and s, respectively.

We now give examples of indecomposable Stone spaces of rank 1 with arbitrary covering dimensions.

THEOREM 5.2. *There exists a Stone space* X *of rank* 1 *with closed subsets* $A_0 \subset A_1 \subset A_2 \subset \cdots$ *such that* i) cov dim $A_n = n$ *for each* $n \geq 0$ *and* ii) A_n *is not the union of countably many relatively open subsets of covering dimension less than* n.

Proof. Let Ω be the first uncountable ordinal, and let Y be the Cantor space 2^{Ω}. We regard Y as the set of subsets of Ω. For each pair of finite disjoint subsets F and G of Ω, let $B(F, G) = \{y \in Y \mid F \subseteq y,\ y \cap G = \emptyset\}$. (The compact, open sets B(F, G) form a base for the topology on Y.) For each nonnegative integer n, let $X_n = \{y \in Y \mid |y| \geq n+1\}$ (where $|y|$ denotes the cardinality of the subset y). Let $X = X_0$, and let $A_n = X \setminus X_{n+1}$ for each $n \geq 0$. Now fix n, and let $B(F, G, n) = B(F, G) \cap A_n$. Then

$$\mathscr{B}_n = \{B(F, G, n) \mid 1 \leq |F| \leq n+1,\ |G| < \infty,\ F \cap G = \emptyset\}$$

is a compact, open base for the topology on A_n.

The sets $B(\{\xi\}, \emptyset, n)$ form a compact, open cover of A_n of order n; therefore cov dim $A_n \leq n$. The proof will be complete once we verify ii). Suppose $\mathscr{U}$ is a countable open cover of A_n. Then some open set $U \in \mathscr{U}$ contains uncountably many singletons $\{\xi\}$ $(\xi \in \Omega)$. We shall show that cov dim $U \geq n$. Let $\mathscr{V}$ be a compact, open cover of U. Clearly we may assume $\mathscr{V} \subseteq \mathscr{B}_n$. For each $\{\xi\} \in U$, choose $V_\xi \in \mathscr{V}$ so that $\{\xi\} \in V_\xi$. Then V_ξ must be of the form $B(\{\xi\}, G_\xi, n)$, where G_ξ is some finite set not containing ξ. Since Y is separable [4, p. 139] and each set $B(\{\xi\}, G_\xi)$ is a nonempty open set in Y, there exists a sequence $\xi_0 < \cdots < \xi_n$ such that $\bigcap_{i=0}^{n} B(\{\xi_i\}, G_{\xi_i}) \neq \emptyset$. The point $x = \{\xi_0, \cdots, \xi_n\}$ is certainly in this intersection. Since $x \in A_n$, we see that $V_{\xi_0} \cap \cdots \cap V_{\xi_n} \neq \emptyset$. ■

Remark. By replacing Ω by $\aleph_k$, we obtain a Stone space $A_{n,k}$, of rank k and covering dimension n, that cannot be expressed as a union of fewer than $\aleph_k$ open subsets of covering dimension less than n. Although the Cantor space $2^{\aleph_n}$ is not necessarily separable, it still has the countable chain condition, which is enough to produce the required sequence $\xi_0 < \cdots < \xi_n$.

PROPOSITION 5.3. *Let* X *be a Stone space of covering dimension* 1. *If* $\check{H}^1(X; \mathbb{Z}_2) = 0$, *then* X *is completely decomposable.*

Proof. Let $\mathscr{U} = \{U_i\colon i \in I\}$ be a compact, open cover of order 1, and assume I is totally ordered. Consider the cochain complex $C^*(\mathscr{U}; \mathbb{Z}_2)$, where

$$C^n(\mathscr{U}; \mathbb{Z}_2) = \prod_{i_0 < \cdots < i_n} \Gamma(U_{i_0 \cdots i_n}, \mathbb{Z}_2).$$

By Theorem 2.1, $H^1(C^*(\mathscr{U}; \mathbb{Z}_2)) = 0$. Define $f \in C^1(\mathscr{U}; \mathbb{Z}_2)$ by letting $f_{ij}(x) = 1$ whenever $i < j$ and $x \in U_{ij}$. Then f is a cocycle, since $C^2(\mathscr{U}; \mathbb{Z}_2) = 0$. Hence there exists $g \in C^0(\mathscr{U}; \mathbb{Z}_2)$ such that $\delta g = f$, that is, $g_i(x) + g_j(x) = 1$ whenever $x \in U_{ij}$. Let $U = \bigcup_{i \in I} g_i^{-1}(0)$ and $V = \bigcup_{I \in I} g_i^{-1}(1)$. Clearly, U and V are paracompact, open sets whose union is X.

Remarks. The converse of Proposition 5.3 is false: the space X_1 of Theorem 4.1 is a counterexample. To show that $H^1(X_1; \mathbb{Z}_2) \neq 0$, it is sufficient to find a continuous map $f\colon U_0 \cap U_1 \to \mathbb{Z}_2$ that cannot be expressed in the form $f_0 + f_1$, where

f_0 and f_1 can be extended continuously to U_0 and U_1, respectively. (See, for example, [2, pp. 219, 220].) Since $U_0 \cap U_1$ is the discrete space $\omega \times \Omega$, we may define $f(n, \alpha) = \varepsilon(n)\varepsilon(\alpha)$, where $\varepsilon(\alpha) = 0$ if α is a limit ordinal and $\varepsilon(\alpha + 1) = \varepsilon(\alpha) + 1$. The verification that f is not of the required form is easy.

Unfortunately, I know of no space satisfying the hypotheses of Theorem 5.3. If we grant the continuum hypothesis, however, we can easily exhibit a Stone space X of cohomological dimension 1 such that $\check{H}^1(X; \mathbb{Z}_2) = 0$. Let X be some non-$\sigma$-compact open subset of the Stone-Čech compactification of the integers. Then, by the countable-chain condition, X must have infinite covering dimension. Therefore $\dim X \geq 1$. By Theorem 2.3 and the continuum hypothesis, $\dim X = 1$. To see that $H^1(X; \mathbb{Z}_2) = 0$, let R be the Boolean ring of subsets of the integers, and let J be the ideal corresponding to the open set X. Then $\mathrm{Ext}^1(J, R) = 0$, since R is self-injective [6, Corollary 24.3]. Therefore $\check{H}^1(X; \mathbb{Z}_2) = 0$, by [10, Theorem 2.1].

Notice that there is no hope of generalizing Proposition 5.3 to higher dimensions, since for $n \geq 2$ the spaces A_n of Theorem 5.2 are indecomposable even though they have trivial n-dimensional cohomology.

Another type of decomposition is possible for every Stone space of finite covering dimension.

THEOREM 5.4. *Let* X *be a Stone space. Then* X *has a dense, open, paracompact subset* D. *Moreover, if* X *has finite covering dimension, then* D *may be chosen so that* $\mathrm{cov\,dim}(X \setminus D) < \mathrm{cov\,dim}\, X$.

Proof. The first statement is trivial: take D to be the union of a maximal family of pairwise disjoint, compact, open sets. Now suppose $\mathrm{cov\,dim}\, X = n < \infty$. By Zorn's lemma, there exists a compact open cover $\mathscr{U}$, maximal with respect to the property that $o(\mathscr{U}) = n$. Let R be the set of points that are contained in precisely $n + 1$ members of $\mathscr{U}$, and let S be the set of isolated points of X. Clearly, $D = R \cup S$ is an open, paracompact subset of X, and $\mathrm{cov\,dim}(X \setminus D) \leq n - 1$. To show that D is dense, suppose to the contrary that $x \notin \overline{D}$, and let N_0 be a compact open neighborhood of x that misses D. Since x is not isolated, there exists a strictly decreasing sequence $N_0 \supset N_1 \supset \cdots$ of compact, open neighborhoods of x. Since $o(\mathscr{U}) < \infty$, some N_k does not belong to $\mathscr{U}$, contrary to the maximality of $\mathscr{U}$.

Suppose X is a completely decomposable Stone space of covering dimension $n \geq 1$. Then X cannot be normal. For if it were, the paracompact open cover $\{U_0, \cdots, U_n\}$ could be shrunk to a closed (paracompact) cover $\{V_0, \cdots, V_n\}$, and it would follow that X is paracompact [5, Footnote 2]. This raises the following question: Are there any normal Stone spaces with finite positive covering dimension? The answer is unknown to the author; but Theorem 5.4 and induction provide an apparently simpler formulation: Is there a normal Stone space X of finite, positive covering dimension with a dense open subset D such that D and $X \setminus D$ are both paracompact? If X is the space A_1 of Theorem 5.2 and $D = A_1 \setminus A_0$, then D and $X \setminus D$ are both discrete (in the relative topology). Of course, A_1 is not normal.

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