SMOOTH HOMOTOPY LENS SPACES

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To R. L. Wilder, with warm appreciation

1. INTRODUCTION

In the following, all manifolds are assumed to be smooth (unless it is otherwise stated) and all actions are differentiable. We are interested in free actions of finite cyclic groups on homotopy spheres.

The existence of free involutions on homotopy spheres has been studied in some detail. In particular, F. Hirzebruch [2] and Orlik and C. P. Rourke [8] proved that every element of the group $\theta^{4k+3}(\partial\pi)$ $(k \geq 1)$ of homotopy spheres that bound π-manifolds admits free involutions. It follows from a result of E. Brieskorn [1] that the same is true of the (possibly) nontrivial element of $\theta^{4k+1}(\partial\pi)$. Since $\theta^{2k}(\partial\pi) = 0$, we again have a free Z_2-action.

If $m > 2$, then clearly only odd-dimensional spheres can admit free Z_m-actions. In Section 2, we define free actions of Z_m and fixed-point-free actions of U(1) on Brieskorn spheres. We use these in Section 3 to prove that *for each prime* p, *every element of* $\theta^{2k+1}(\partial\pi)$ $(k > 1)$ *admits a free action of* Z_p. This contrasts with the fact that not every element of $\theta^{2k+1}(\partial\pi)$ admits a *free* circle action.

In Section 4 we compare our actions with those obtained by J. Milnor [4], by D. Montgomery and C. T. Yang [5], and by C. N. Lee [3], and we show that some are definitely distinct from those previously known. In Section 5, we describe the Brieskorn spheres as branched finite cyclic coverings of the standard sphere, branched along a Brieskorn variety of codimension 2. This is used in Section 6 to determine the *homotopy types* of the orbit spaces of the Z_m-actions of Section 2. In Section 7 we determine their *stable tangent bundle* and *characteristic classes*, and in Section 8 we consider some normal bundles. In the 3-dimensional analogue we obtain free actions on homology spheres. A complete classification of the arising homology lens spaces is given in Section 9.

I am indebted to F. Hirzebruch for pointing out Lemma 2 and showing how it completes the argument of Theorem 1, to F. Raymond for helpful suggestions, and to R. Lee for stimulating conversations.

2. ACTIONS ON BRIESKORN SPHERES

Recall the variety

$$V(a) = \{z\colon z \in \mathbb{C}^{n+1},\ z_0^{a_0} + \cdots + z_n^{a_n} = 0\}$$

considered by E. Brieskorn [1]. Here $a = \{a_0, \cdots, a_n\}$ is a set of integers $(a_j \geq 2$ for each j). V(a) has an isolated singularity at the origin. Its intersection with the unit sphere in $\mathbb{C}^{n+1}$,

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$$K(a) = \left\{ z\colon z \in \mathbf{C}^{n+1},\ z_0^{a_0} + \cdots + z_n^{a_n} = 0,\ \sum_{i=0}^{n} z_i \bar{z}_i = 1 \right\},$$

is a smooth $(2n-1)$-manifold. Following [1], we let $G_a = G(a_0, \cdots, a_n)$ denote the graph with $n+1$ vertices with weights $a_0, \cdots, a_n$. Two vertices in G_a with weights a_i and a_j are connected if the greatest common divisor (a_i, a_j) is greater than 1.

THEOREM [1]. *For* $n > 2$, $K(a)$ *is a homotopy sphere if and only if one of the following conditions is satisfied:*

(i) G_a *has at least two isolated vertices,*

(ii) G_a *has one isolated vertex, and at least one of its components consists of an odd number of vertices, each with even weight, and with* $(a_i, a_j) = 2$ *for* $i \neq j$.

For $n = 2$, the condition (i) is necessary and sufficient in order that $K(a)$ be an integral homology sphere.

Whenever we wish to emphasize that $K(a)$ is a homotopy sphere $(n > 2)$ or homology sphere $(n = 2)$, we shall write $\Sigma(a)$.

Let $a = \{a_0, \cdots, a_n\}$ be a set of positive integers such that $K(a)$ is a homotopy sphere. (This assumption is not necessary for defining the action below, but here we are interested only in the case where $K(a)$ is a sphere.)

Let m be a positive integer, relatively prime to each a_j. Define $q_j \geq 1 \pmod m$ by the equation

$$q_j a_j \equiv 1 \pmod m \qquad (j = 0, \cdots, n).$$

Define an action of Z_m on $\mathbf{C}^{n+1}$ by

$$\alpha(z_0, \cdots, z_n) = (\alpha^{q_0} z_0, \cdots, \alpha^{q_n} z_n),$$

where α is a primitive mth root of unity considered as the preferred generator of Z_m. Clearly, S^{2n+1}, $V(a)$, and $\Sigma(a)$ are invariant under this action, which fixes the origin and is free otherwise. Let

$$Q_m^{2n-1}(a_0, \cdots, a_n) = \Sigma^{2n-1}(a_0, \cdots, a_n)/Z_m$$

be the orbit space of the action. Note that $Q_m^{2n-1}(a)$ is a submanifold (of codimension 2) of the lens space $L_m^{2n+1}(a_0, \cdots, a_n)$, the latter being the orbit space of the above action on S^{2n+1}. For notation, see Milnor [4].

The order of the a_j is immaterial; hence, if $(a_0', \cdots, a_n')$ is a permutation of $(a_0, \cdots, a_n)$, then $Q_m(a_0', \cdots, a_n')$ is diffeomorphic to $Q_m(a_0, \cdots, a_n)$.

An alternative approach is to let c denote the least common multiple of $\{a_0, \cdots, a_n\}$ and to define the integers c_j by the equations

$$a_j c_j = c \qquad (j = 0, \cdots, n).$$

We can then define an action of $U(1)$ on $\mathbf{C}^{n+1}$ by letting each $g \in U(1)$ act according to the formula

$$g(z_0, \cdots, z_n) = (g^{c_0} z_0, \cdots, g^{c_n} z_n).$$

This action also leaves S^{2n+1}, V(a), and $\Sigma(a)$ invariant. Its only fixed point is the origin; but it has orbits with nontrivial stability groups.

Now, if m is relatively prime to the integers $a_0, \cdots, a_n$, then we can let β equal a primitive mth root of unity and define an action of $Z_m \subset U(1)$ by

$$\beta(z_0, \cdots, z_n) = (\beta^{c_0} z_0, \cdots, \beta^{c_n} z_n).$$

Clearly,

$$c_j \equiv cq_j \pmod m \qquad (j = 0, \cdots, n);$$

thus the only difference between the actions of α and β is the choice of preferred generator. If we choose β so that $\beta^c = \alpha$, then the actions are clearly the same.

3. FREE Z_p-ACTIONS ON $\theta^{2n-1}(\partial\pi)$

Recall that $\theta^{4k+1}(\partial\pi)$ is a subgroup of Z_2, while, except for a possible factor 2, the order of $\theta^{4k-1}(\partial\pi)$ equals

$$2^{2k-2}(2^{2k-1} - 1) \cdot \text{numerator}\left(\frac{4B_k}{k}\right),$$

where B_k is the kth Bernoulli number. The order of $\theta^{4k-1}(\partial\pi)$ is never divisible by 3 (see [2]).

THEOREM 1. *Every odd-dimensional homotopy sphere* Σ^{2n-1} $(n > 2)$ *that bounds a parallelizable manifold admits a free* Z_p*-action for each prime* p.

LEMMA 1. *If* m *is relatively prime to the order of* θ^{2n-1}, *then there is a free* Z_m*-action on each element of* θ^{2n-1}.

The lemma is essentially due to Lee [3]. To prove it, let S^{2n-1} be the standard sphere with a standard linear free Z_m-action T. For any homotopy sphere Σ^{2n-1}, let $[\Sigma]$ denote the element it represents in θ^{2n-1}. The universal cover of $(S^{2n-1}/T) \# \Sigma^{2n-1}$ represents $m[\Sigma]$. Since m is relatively prime to the order of θ^{2n-1}, every element is obtained this way.

By Brieskorn [1], the (possibly) nontrivial element of $\theta^{4k+1}(\partial\pi)$ is diffeomorphic to

$$\Sigma^{4k+1}(2, \cdots, 2, 3).$$

The involution

$$T(z_0, \cdots, z_{2k+1}) = (-z_0, \cdots, -z_{2k}, z_{2k+1}) \qquad .$$

is free on Σ. Since the order of $\theta^{4k+1}(\partial\pi)$ is at most 2, the theorem is established in this dimension.

Now let us turn to the (4k - 1)-spheres. Let Σ_1^{4k-1} denote the Milnor sphere, that is, let Σ_1^{4k-1} bound a π-manifold with index 8.

LEMMA 2. *If* $\Sigma^{4k-1} \in \theta^{4k-1}(\partial\pi)$ *and* $[\Sigma] = j[\Sigma_1]$, *then*

$$\Sigma^{4k-1} = \Sigma^{4k-1}(2, \cdots, 2, 3, 6j - 1)$$

and

$$\Sigma^{4k-1} = \Sigma^{4k-1}(2, \cdots, 2, 3, 6j+1).$$

The lemma follows from direct computations according to [1, Section 7].

In order to complete the proof of the theorem, note that for $(4k-1)$-spheres and $p = 2$, the theorem was established in [2] and [8]. For $p = 3$, Lemma 1 is applicable. If $p \neq 2, 3$, then p is prime either to $6j - 1$ or to $6j + 1$, and we can use the construction of Section 2.

COROLLARY 1. *Every odd-dimensional homotopy sphere* Σ^{2n-1} $(n > 2)$ *that bounds a parallelizable manifold admits a free* Z_{p^r}*-action, for each odd prime* p *and each positive integer* r.

Furthermore, if there exists an orthogonal representation

$$h: Z_p \to O(n)$$

such that the induced homomorphism

$$h_*: Wh(Z_p) \to R^+$$

is nontrivial, then *every odd-dimensional homotopy sphere that bounds a parallelizable manifold admits infinitely many distinct free* Z_p*-actions* (see Milnor [4]).

It was pointed out by C. N. Lee [3] that Lemma 1 can be used to obtain free Z_m-actions on homotopy spheres that do not bound parallelizable manifolds.

Examples. Since θ^9 has order 8, there is a free Z_{2k+1}-action for each $k \geq 1$, on every element of θ^9. The same is true for θ^{17}, since it is a group of order 16.

For $\theta^{15} = Z_2 + Z_{8128}$, every 15-sphere admits a free action of Z_p for all odd primes p. Also, by an example of Bredon, at least half of the spheres not in bP_{16} admit free involutions.

Montgomery and Yang [5] have shown that there are elements of $\theta^7 = \theta^7(\partial\pi)$ that admit no free circle actions. Thus our theorem yields the following result.

COROLLARY 2. *A manifold may admit free actions of finite groups of arbitrarily high order without admitting a free action by a compact connected Lie group.*

Note that we get free Z_m-actions for many integers m that are not powers of primes. A more detailed study of the different possible Brieskorn varieties diffeomorphic to the same sphere in the spirit of Lemma 2 may suffice to prove the following conjecture.

Conjecture. Every odd-dimensional homotopy sphere Σ^{2n-1} $(n > 2)$ that bounds a parallelizable manifold admits a free Z_m-action for each integer $m \geq 2$.

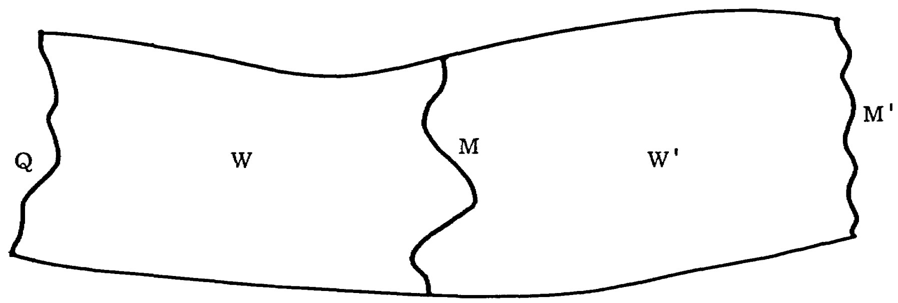
4. COMPARISON WITH KNOWN FREE Z_p-ACTIONS

Some of the Z_p-actions constructed above are new. Recall that the definition of a lens space is that of the quotient of S^{2n+1} by a standard orthogonal Z_p-action. Milnor [4] proved that if two lens spaces are h-cobordant, then they are s-cobordant. He used this to obtain new free Z_p-actions as follows. Let M be a manifold that is h-cobordant to a lens space in such a way that the Reidemeister torsion of the h-cobordism is nonzero (in particular, this is possible for every prime p greater than 3). Then the universal cover of M is S^{2n+1}, and the action of Z_p is not equivalent to a standard action.

Of course, all our Z_p-actions on $\Sigma^{2n+1} \neq S^{2n+1}$ are differentiably distinct from the examples of Milnor. Even if we consider all translates of the Milnor actions in the sense of Lemma 1, we can choose a prime p that divides the order of $\theta^{4k-1}(\partial\pi)$, and our Z_p-actions on $\Sigma_1^{4k-1}, \cdots, \Sigma_{p-1}^{4k-1}$ are not of this type.

In dimension 7, Montgomery and Yang [5] obtained free Z_p-actions different from the Milnor examples by considering the restrictions of a free S^1-action on $\Sigma^7 \neq S^7$. If we let $p = 7$ and apply Lemma 1 to the standard actions, to the Milnor examples, and to the Montgomery-Yang examples, we see that they give no Z_7-action on Σ_2^7 and Σ_5^7, for example. Thus some of our actions are still different.

In fact, it is easy to see that the orbit spaces of these examples are not even PL h-cobordant to a Milnor homotopy lens space. Indeed, suppose Q^7 is the orbit space of one of the above actions, and there is a PL h-cobordism W with a Milnor homotopy lens space M^7. Let τ be the torsion of this h-cobordism. Construct W', a smooth h-cobordism



from M to M' with torsion $-\tau$. Then $W \cup W'$ is a trivial PL h-cobordism between the smooth manifolds Q and M'. Note that M' is again a Milnor homotopy lens space. Hence M' is a different smoothing of Q, and in dimension 7 this means that $Q \simeq M' \# \Sigma$ for some $\Sigma \in \theta^7 = \theta^7(\partial\pi)$. This leads to the contradiction that the action is a "translate" of a Milnor example.

5. BRANCHED CYCLIC COVERINGS

The following construction may be of some interest on its own right. Let

$$K^{2n-1}(a) = \left\{ (z_0, z_1, \cdots, z_n) : z_0^{a_0} + \cdots + z_n^{a_n} = 0, \sum_{i=0}^{n} z_i \bar{z}_i = 1 \right\}.$$

Define a map

$$\Phi: K^{2n-1}(a) \to S^{2n-1}$$

by $\Phi(z_0, \cdots, z_n) = (\rho_0 z_1, \rho_0 z_2, \cdots, \rho_0 z_n)$, where $|z_0| = r_0$ and $\rho_0 = (1 - r_0^2)^{-1/2}$. Clearly, Φ is an a_0-fold branched cyclic covering of S^{2n-1}. The branching occurs along the Brieskorn variety

$$K^{2n-1}(a) \cap \{z_0 = 0\} = K^{2n-3}(a_1, \cdots, a_n).$$

Of course, $K^{2n-1}(a)$ is similarly the a_i-fold branched covering over S^{2n-1}, branched along $K(a) \cap \{z_i = 0\}$.

As an example, we may obtain the 11-dimensional Milnor sphere $\Sigma^{11}(2, 2, 2, 2, 2, 3, 5)$ as a 5-fold cyclic cover of S^{11}, branched along the 9-dimensional Kervaire sphere $\Sigma^9(2, 2, 2, 2, 2, 3)$.

Note that this is a straightforward generalization of the 3-dimensional case; for example, the well-known Poincaré space $\Sigma^3(2, 3, 5)$ is a cyclic 5-fold covering of S^3, branched along the torus knot $\Sigma^1(2, 3)$.

6. HOMOTOPY TYPE OF $Q_m^{2n-1}(a)$

We can define an action of $U(1)$ (not necessarily effective) on $\mathbb{C}^n$ by

$$g(z_1, \cdots, z_n) = (g^{c_1} z_1, \cdots, g^{c_n} z_n),$$

where the c_j are defined as in Section 2. With respect to this action, Φ is equivariant.

Similarly, if we define an action of Z_m on $\mathbb{C}^n$ by

$$\alpha(z_1, \cdots, z_n) = (\alpha^{q_1} z_1, \cdots, \alpha^{q_n} z_n),$$

then Φ commutes with the action.

Now suppose $K(a)$ is a homotopy sphere, and $n > 2$. Then we have the commutative diagram

$$\begin{array}{ccc} \Sigma^{2n-1}(a) & \xrightarrow{\Phi} & S^{2n-1} \\ \pi \downarrow & & \downarrow \pi \\ Q_m^{2n-1}(a) & \xrightarrow{\phi} & L_m^{2n-1}(a_1, \cdots, a_n) \end{array},$$

where $\deg \phi = \deg \Phi = a_0$.

Following P. Olum [6], we define a map

$$\Psi: S^{2n-1} \to S^{2n-1}$$

by $\Psi(z_j) = z_j$ $(j = 2, \cdots, n)$, and if $z_1 = r_1 e^{i\theta_1}$, we let

$$\Psi(r_1) = r_1, \quad \Psi(\theta_1) = q_0 \theta_1.$$

Here Ψ covers a map of lens spaces

$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{\Psi} & S^{2n-1} \\ \pi \downarrow & & \downarrow \pi \\ L_m^{2n-1}(a_1, \cdots, a_n) & \xrightarrow{\psi} & L_m^{2n-1}(a_0 a_1, a_2, \cdots, a_n) \end{array},$$

and $\deg \psi = \deg \Psi = q_0$.

Thus the composition $h = \psi \circ \phi$

$$h: Q_m^{2n-1}(a_0, a_1, \cdots, a_n) \to L_m^{2n-1}(a_0 a_1, a_2, \cdots, a_n)$$

is a map of degree $a_0 q_0 \equiv 1 \pmod{m}$. Using a theorem of Olum [6], we now obtain the following result.

THEOREM 2. *Let* $n > 2$.

(i) *There exists a homotopy equivalence*

$$Q_m^{2n-1}(a_0, \cdots, a_n) \to L_m^{2n-1}(b_1, \cdots, b_n)$$

preserving orientation and preferred generator if and only if

$$a_0 \cdots a_n \equiv b_1 \cdots b_n \pmod{m}.$$

(ii) *There exists a homotopy equivalence*

$$Q_m^{2n-1}(a_0', \cdots, a_n') \to Q_m^{2n-1}(a_0, \cdots, a_n)$$

preserving orientation and preferred generator if and only if

$$a_0' \cdots a_n' \equiv a_0 \cdots a_n \pmod{m}.$$

(iii) *There exists a homotopy equivalence*

$$Q_m^{2n-1}(a_0, \cdots, a_n) \to L_m^{2n-1}(b_1, \cdots, b_n)$$

if and only if for some integer k *with* $(k, m) = 1$,

$$a_0 \cdots a_n \equiv \pm k^n b_1 \cdots b_n.$$

(iv) *There exists a homotopy equivalence*

$$Q_m^{2n-1}(a_0', \cdots, a_n') \to Q_m^{2n-1}(a_0, \cdots, a_n)$$

if and only if for some integer k *with* $(k, m) = 1$,

$$a_0' \cdots a_n' \equiv \pm k^n a_0 \cdots a_n.$$

7. THE STABLE TANGENT BUNDLE

The imbedding $\Sigma^{2n-1}(a) \subset \mathbb{C}^{n+1} = R^{2n+2}$ is equivariant with respect to the action of Z_m (with generator α) defined in Section 2. The representation

$$\zeta\colon Z_m \to O(2n+2)$$

is the sum of the representations

$$\zeta(q_j)\colon Z_m \to SO(2),$$

$$\zeta(q_j)\alpha = \begin{vmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{vmatrix},$$

where $\gamma = 2\pi q_j/m$, and where the q_j are relatively prime to m.

The action of Z_m on R^{2n+2} is precisely the action defined by

$$\zeta(q_0) \oplus \cdots \oplus \zeta(q_n).$$

This action defines a principal Z_m-bundle ξ with total space $E_\xi = \Sigma^{2n-1}(a)$ and orbit space $B_\xi = Q_m^{2n-1}(a)$.

Let $f\colon Q \to B(Z_m)$ classify ξ. We denote by ξ_j the 2-plane bundle over Q associated with the representation $\zeta(q_j)$, that is, the pullback of the universal 2-plane bundle by the composite $\lambda_j \circ f$,

$$Q \xrightarrow{f} B(Z_m) \xrightarrow{\lambda_j} B(SO(2)),$$

where λ_j is induced by $\zeta(q_j)$.

The normal bundle of $\Sigma^{2n-1}(a)$ in R^{2n+2} is trivial, since $\Sigma^{2n-1}(a)$ is already in R^{2n+1} and its normal 2-plane bundle there is clearly trivial. Moreover, the bundle has an equivariant cross-section.

Denote by $\tau(M)$ the tangent bundle of M, and by θ^n the trivial n-plane bundle. The following proposition now follows from a theorem of R. H. Szczarba [10].

THEOREM 3. $\tau(Q_m^{2n-1}) \oplus \theta^3 = \xi_0 \oplus \xi_1 \oplus \cdots \oplus \xi_n$.

The preferred generator α of $H_1(Q; Z) = Z_m$ determines a preferred generator v for the dual group $H^2(Q; Z)$, and if u is its reduction (mod 2), we obtain the following total Pontrjagin and Stiefel-Whitney classes.

THEOREM 4.

$$p(Q_m^{2n-1}) = \prod_{j=0}^{n} (1 + q_j^2 v^2),$$

$$w(Q_m^{2n-1}) = \prod_{j=0}^{n} (1 + q_j u) \pmod 2 .$$

8. SOME NORMAL BUNDLES

Recall that we have the codimension-2 imbedding

$$i\colon Q_m^{2n-1}(a_0, \cdots, a_n) \to L_m^{2n+1}(a_0, \cdots, a_n).$$

We claim that if η is the normal 2-plane bundle of this imbedding, then η is trivial.

Indeed, by [10],

$$\tau(L_m^{2n+1}) \oplus \theta^1 = \xi_0 \oplus \xi_1 \oplus \cdots \oplus \xi_n;$$

hence

$$\tau(Q) \oplus \theta^3 = i^* \tau(L) \oplus \theta^1 = \tau(Q) \oplus \eta \oplus \theta^1,$$

and therefore η is stably trivial and hence trivial.

Note also that this implies that if ν_M is the stable normal bundle of M, then $\nu_Q = i^* \nu_L$.

This relation never holds for an imbedded lens space of codimension 2, $L_m^{2n-1} \subset L_m^{2n+1}$.

There is another natural imbedding of codimension 2,

$$Q_m^{2n-1}(a_0, \cdots, a_n) \subset Q_m^{2n+1}(a_0, \cdots, a_n, a_{n+1}),$$

for any integer a_{n+1} that is relatively prime to m and for which $a = \{a_0, \cdots, a_{n+1}\}$ again satisfies the Brieskorn conditions. These conditions are satisfied, for example, in the case $Q_m^{4k-3}(2, \cdots, 2, 3) \subset Q_m^{4k-1}(2, \cdots, 2, 3, 5)$. The normal bundle of this imbedding has Euler class (first Chern class) equal to q_{n+1}, where $q_{n+1} a_{n+1} \equiv 1 \pmod{m}$.

Thus the branched coverings constructed in Section 5 give rise to a commutative diagram

$$\begin{array}{ccc} Q_m^{2n+1}(a_0, \cdots, a_n, a_{n+1}) & \xrightarrow{\phi_{2n+1}} & L_m^{2n+1}(a_1, \cdots, a_n, a_{n+1}) \\ \uparrow i_Q & & \uparrow i_L \\ Q_m^{2n-1}(a_0, \cdots, a_n) & \xrightarrow{\phi_{2n-1}} & L_m^{2n-1}(a_1, \cdots, a_n) \end{array} ,$$

where i_L is the natural imbedding.

Remark. It would be interesting to know whether a homotopy equivalence

$$h: Q_m^{2n-1}(a) \to L_m^{2n-1}$$

may be chosen so that h is a *tangential equivalence.* Of course, even this would not suffice to determine the normal invariants of $Q_m^{2n-1}(a)$.

Added in proof. $Q_5^7(2, 2, 2, 3, 13)$ is not tangentially equivalent to any lens space.

9. THE 3-DIMENSIONAL HOMOLOGY LENS SPACES

For $n = 2$, the Brieskorn variety

$$K^3(a_0, a_1, a_2) = \left\{ (z_0, z_1, z_2): z_0^{a_0} + z_1^{a_1} + z_2^{a_2} = 0, \sum_{i=0}^{2} z_i \bar{z}_i = 1 \right\}$$

is a closed 3-manifold, which according to Section 2 admits an effective circle action. In [7], the actions of $SO(2) = U(1)$ on 3-manifolds are classified equivariantly and topologically. The question of fitting an arbitrary $K^3(a)$ into this classification will be treated in a subsequent paper. Since in the present case $K^3(a)$ is a homology sphere (Poincaré space), this is not difficult.

According to Brieskorn, $K^3(a)$ is a homology sphere $\Sigma^3(a)$ if and only if a_0, a_1, a_2 are relatively prime. Thus the $U(1)$-action on $\Sigma^3(a)$ is

$$g(z_0, z_1, z_2) = (g^{a_1 a_2} z_0, g^{a_0 a_2} z_1, g^{a_0 a_1} z_2).$$

This action is fixed-point free. Since the homology sphere $\Sigma^3(a)$ is orientable, the $U(1)$-action has no orbits reversing the local orientation, and the orbit space is orientable. By [7], it is a weighted 2-manifold. A simple homology argument shows that it must be a weighted 2-sphere.

There are 3-orbits with finite stability groups. For $z_i = 0$, the torus knot $K^1(a_{i+1}, a_{i+2})$ has stability group Z_{a_i} $(i = 0, 1, 2)$ (mod 3). Thus our 3-manifold is described equivariantly (see [7]) as

$$\Sigma^3(a_0, a_1, a_2) = \{b; (o, 0, 0, 0); (a_0, \beta_0), (a_1, \beta_1), (a_2, \beta_2)\}.$$

The integers $b, \beta_0, \beta_1, \beta_2$ $(0 < \beta_i < a_i, \ (a_i, \beta_i) = 1, \ i = 0, 1, 2)$ describe the circle action in the neighborhoods of a principal orbit and of the three nonprincipal orbits. In our case, they may be determined from purely algebraic facts. The order of $H_1(\Sigma(a); Z)$ equals

$$p = |b a_0 a_1 a_2 + a_0 a_1 \beta_2 + a_0 \beta_1 a_2 + \beta_0 a_1 a_2|,$$

and since $\Sigma^3(a)$ is a homology sphere, $p = \pm 1$. It turns out that there are exactly two sets of solutions, corresponding to the two orientations of $\Sigma^3(a)$. Moreover, $b = -1$ or $b = -2$, and if $\{-2, \beta_0, \beta_1, \beta_2\}$ is one solution, then the other is $\{-1, (a_0 - \beta_0), (a_1 - \beta_1), (a_2 - \beta_2)\}$.

It can be shown that the usual orientation inherited from $\mathbb{C}^3$ yields $b = -2$, but we shall be satisfied with a classification up to orientation.

The Poincaré spaces in question were first treated by Seifert [9], who also determined the orbit space of a free Z_m-action $(Z_m \subset U(1))$ on $\Sigma^3(a)$.

Specifically, let m be relatively prime to a_0, a_1, a_2. The orbit space of the free Z_m-action generated by β in Section 2 is

$$Q^3_m(a) = \{mb; (o, 0, 0, 0), (a_0, m\beta_0), (a_1, m\beta_1), (a_2, m\beta_2)\}.$$

In order to normalize this description (see [9], [7]), let

$$m\beta_i = r_i a_i + \delta_i \qquad (0 < \delta_i < a_i, \ i = 0, 1, 2),$$

and let $d = mb + r_0 + r_1 + r_2$. Then

$$Q^3_m(a_0, a_1, a_2) = \{d; (o, 0, 0, 0), (a_0, \delta_0), (a_1, \delta_1), (a_2, \delta_2)\}.$$

We apply [9, Satz 12] and [7, Theorem 4] to obtain the following result.

THEOREM 5. *Let* $Q = Q^3_m(a_0, a_1, a_2)$ *and* $Q' = Q^3_m(a'_0, a'_1, a'_2)$ *be homology lens spaces as above. Then the following statements are equivalent:*

(i) Q *and* Q' *are equivariantly diffeomorphic.*

(ii) Q *and* Q' *are diffeomorphic.*

(iii) Q *and* Q' *are homeomorphic.*

(iv) $\{a'_0, a'_1, a'_2\}$ *is a permutation of* $\{a_0, a_1, a_2\}$.

By [7], we may add the following assertions.

1) The induced $U(1)$-action on $Q^3_m(a)$ is the only circle action admitted by $Q^3_m(a)$.

2) $Q^3_m(2, 3, 5)$, the orbit space of a free Z_m-action on the well-known Poincaré space $\Sigma^3(2, 3, 5)$, is the only one of our homology lens spaces with finite fundamental group. Its universal cover is S^3.

3) All other $Q^3_m(a)$ are $K(\pi, 1)$'s with infinite fundamental group.

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