DISTRIBUTIVE LOCAL NOETHER LATTICES

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1. INTRODUCTION

In a recent paper, R. P. Dilworth [2] introduced the concept of a Noether lattice as an abstraction of the concept of the lattice of ideals of a Noetherian ring. A Noether lattice is a modular multiplicative lattice that satisfies the ascending chain condition and in which every element is a join of elements called principal elements. The principal elements are characterized by a pair of identities that are satisfied by the principal ideals of a ring. The usual ring-theoretic definitions of the terms *local, regular, dimension,* and *rank* carry over directly to Noether lattices. In a recent paper [1], the author showed that a distributive regular local Noether lattice of dimension n is isomorphic to RL_n, the sublattice of the lattice of ideals of $F[x_1, \cdots, x_n]$ generated by the principal ideals $(x_1), \cdots, (x_n)$ under multiplication and join.

The purpose of this paper is to describe the structure of distributive local Noether lattices. Loosely this description states that each distributive local Noether lattice L is obtained from one of the lattices RL_n by identification of some of the principal elements of RL_n with an equivalence relation that preserves join, multiplication, and cancellation in nonzero products, and by the extension of this equivalence relation to all of RL_n. The equivalence classes of RL_n modulo this relation form a lattice isomorphic to L.

Section 2 of this paper contains a characterization of principal elements in a modular multiplicative lattice, which, though known, has not yet appeared in the literature.

We use the notation and terminology of [1]. In particular, E, F, H, K, and N denote principal elements. Definitions given in [1] will not be repeated here.

2. A CHARACTERIZATION OF PRINCIPAL ELEMENTS

Our first theorem shows that in the case of a modular lattice, the defining equations for principal elements can be simplified.

THEOREM 1. *Let* L *be a modular multiplicative lattice. An element* E *of* L *is principal if and only if*

(2.1) $$B \wedge E = (B:E)E \quad \textit{for all } B \in L$$

and

(2.2) $$(BE):E = B \vee 0:E \quad \textit{for all } B \in L.$$

Proof. By Corollaries 3.1 and 3.2 of [2], principal elements of L satisfy (2.1) and (2.2).

We show first that an element E satisfying (2.1) and (2.2) is join-principal, in other words, that

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$$(A \vee BE) : E = A : E \vee B \quad \text{for all } A, B \in L.$$

The proof of the following set of equations uses (2.1), (2.2), the modular law, the join-distributivity of multiplication, and the fact that $0 : Y \leq X : Y$ for all $X, Y \in L$.

$$\begin{aligned}(A \vee BE) : E &= (A \vee BE) : E \vee 0 : E = \{[(A \vee BE) : E]E\} : E \\ &= [(A \vee BE) \wedge E] : E = [(A \wedge E) \vee BE] : E = [(A : E)E \vee BE] : E \\ &= [(A : E \vee B)E] : E = A : E \vee B \vee 0 : E = A : E \vee B.\end{aligned}$$

Now we show that if E satisfies (2.1) and (2.2), then E is meet-principal, in other words, that

$$(A \wedge B : E)E = AE \wedge B \quad \text{for all } A, B \in L.$$

The proof of the following set of equations uses the same facts as the proof of the previous set, and in addition it uses the fact that $(X \wedge Y) : Z = X : Z \wedge Y : Z$ for all $X, Y, Z \in L$.

$$\begin{aligned}AE \wedge B &= AE \wedge E \wedge B = [(AE \wedge B) : E]E = [(AE \wedge E \wedge B) : E]E \\ &= \{[AE \wedge (B : E)E] : E\}E = \{(AE) : E \wedge [(B : E)E] : E\}E \\ &= [(A \vee 0 : E) \wedge (B : E \vee 0 : E)]E = [(A \vee 0 : E) \wedge B : E]E \\ &= [(A \wedge B : E) \vee 0 : E]E = (A \wedge B : E)E \vee 0 = (A \wedge B : E)E.\end{aligned}$$

3. THE BASIC STRUCTURE THEOREM

The distributive law imposes a very strong restriction on the structure of local Noether lattices, as the next two lemmas show.

LEMMA 1. *Let* L *be a distributive local Noether lattice, and let* $A \in L$. *Then any two minimal representations of* A *as a join of principal elements differ only in order, that is, they use the same principal elements.*

Proof. By Lemma 2.2 of [1], the principal elements of L are precisely the join-irreducible elements. It is well known that in a distributive lattice, two minimal representations of an element as a finite join of join-irreducible elements differ only in order.

LEMMA 2. *Let* L *be a distributive local Noether lattice, and let* $M = E_1 \vee \cdots \vee E_n$ *be a minimal representation of the maximal element as a join of principal elements. Then each nonzero proper principal element of* L *is a product of powers of the elements* E_i.

Proof. Let $E \neq I$ be a nonzero principal element of L. Then

$$E = E \wedge M = E \wedge (E_1 \vee \cdots \vee E_n) = (E \wedge E_1) \vee \cdots \vee (E \wedge E_n),$$

which implies that $E = E \wedge E_k$ for some k by Lemma 2.2 of [1]. Thus $E \leq E_k$. Let i(k) be the largest integer such that $E \leq E_k^{i(k)}$. Then

$$E = (E : E_k^{i(k)})E_k^{i(k)},$$

by condition (2.1). Lemma 2.2 of [1] implies that $E = FE_k^{i(k)}$ for some principal element $F < E : E_k^{i(k)}$. But $F \not\leq E_k$, since $E \not\leq E_k^m$ for every m greater than $i(k)$. Thus $F = I$ or $F \leq E_j$ for some $j \neq k$. If $F = I$, there is nothing more to do. If $F \neq I$, we apply the same process to F as we applied to E, and we obtain the equation

$$E = H E_j^{i(j)} E_k^{i(k)},$$

where $i(j)$ is the largest integer such that $F \leq E_j^{i(j)}$. Iteration of this process yields the equation

$$E = K E_1^{i(1)} E_2^{i(2)} \cdots E_n^{i(n)}$$

with $K \not\leq E_m$ $(m = 1, 2, \cdots, n)$. It follows that $K = I$, and the lemma is proved.

From Lemma 2 we see that it is possible to define a map from RL_n onto a distributive local Noether lattice whose maximal element has a minimum representation as a join of n principal elements. The next theorem shows that this allows us to characterize distributive local Noether lattices in a concrete manner.

THEOREM 2. *Let* L *be a distributive local Noether lattice. Then there exist an integer* n, *an equivalence relation* θ *on* RL_n, *and an equivalence relation* σ *on the set of principal elements of* RL_n *such that the equivalence classes of* RL_n *modulo* θ *form a Noether lattice isomorphic to* L, *and such that the following conditions are satisfied.*

(1) $H_1 \vee \cdots \vee H_r \;\theta\; K_1 \vee \cdots \vee K_n$ *if and only if for each* $i \leq r$ *there exist a principal element* N *in* RL_n *and an integer* $j(i) \leq m$ *such that* $H_i \;\theta\; N \leq K_{j(i)}$, *and for each* $j \leq m$ *there exist a principal element* N' *in* RL_n *and an integer* $i(j) \leq r$ *such that* $K_j \;\theta\; N' \leq H_{i(j)}$.

(2) $H \;\sigma\; N$ *implies* $HK \;\sigma\; NK$.

(3) $HK \equiv 0$ (*modulo* σ) *and* $HK \;\sigma\; NK$ *imply* $H \;\sigma\; N$.

(4) $X \;\theta\; I$ *implies* $X = I$.

Conversely, if θ *is an equivalence relation on* RL_n *and* σ *is an equivalence relation on the set of principal elements of* RL_n, *and if* θ *and* σ *satisfy conditions* (1) *to* (4), *then the set of equivalence classes of* RL_n *modulo* θ *forms a Noether lattice.*

Proof. Suppose that L is a distributive local Noether lattice. Let

$$M = E_1 \vee \cdots \vee E_n$$

be a minimal representation of the maximal element of L. By Lemma 3.2, each element of L has a representation as a join of products of powers of the elements E_i. Two elements of RL_n are equal only if they are joins of the same principal elements, by Lemma 3.1. Since, by Lemma 3.2 and by residuation, two principal elements of RL_n are equal only if they are the same product of powers of the elements (x_i), it is possible to define a map ϕ from RL_n into L by defining $\phi[(x_i)] = E_i$ and then extending this map to all of RL_n according to the rules

$$\phi(AB) = \phi(A)\phi(B), \qquad \phi(A \vee B) = \phi(A) \vee \phi(B).$$

Define an equivalence relation θ on RL_n by the rule that $A\ \theta\ B$ if and only if $\phi(A) = \phi(B)$. Let $\langle A \rangle$ denote the set of all elements of RL_n equivalent to A modulo θ. Define

$$\bigvee \langle A \rangle = \bigvee \{A' \mid A'\ \theta\ A\}, \tag{3.1}$$

$$\langle A \rangle \leq \langle B \rangle \quad \text{if and only if } A \leq \bigvee \langle B \rangle, \tag{3.2}$$

$$\langle A \rangle \vee \langle B \rangle = \langle A \vee B \rangle, \tag{3.3}$$

$$\langle A \rangle \wedge \langle B \rangle = \left\langle \left(\bigvee \langle A \rangle \right) \wedge \left(\bigvee \langle B \rangle \right) \right\rangle, \quad \text{and} \tag{3.4}$$

$$\langle A \rangle \langle B \rangle = \langle AB \rangle. \tag{3.5}$$

It is obvious that condition (3.2) defines a partial ordering on the set L' of equivalence classes modulo θ. Since $\bigvee \langle X \rangle\ \theta\ X$ for all X in RL_n, and since $X \leq Y$ implies $\bigvee \langle X \rangle \leq \bigvee \langle Y \rangle$ for all X and Y in RL_n, the relations (3.3) and (3.4) give the meet and join relative to the partial ordering defined in (3.2) for each pair of elements in L'. With the multiplication given in (3.5), L' is a complete multiplicative lattice satisfying the ascending chain condition.

Consider the map $\phi'\colon L' \to L$ defined by $\phi'(\langle A \rangle) = \phi(A)$. By the definition of ϕ' and θ, the relation $\phi'(\langle A \rangle) = \phi'(\langle B \rangle)$ implies $\phi(A) = \phi(B)$, which means that $\langle A \rangle = \langle B \rangle$. Thus ϕ' is one-to-one. If X is in L, then $X = \phi(Y)$ for some Y in RL_n, which implies that $X = \phi'(\langle Y \rangle)$. Therefore ϕ' is onto. Since

$$\langle A \rangle \vee \langle B \rangle = \langle A \vee B \rangle \quad \text{and} \quad \langle A \rangle \langle B \rangle = \langle AB \rangle,$$

ϕ' preserves the partial order and multiplication of L'. Now assume that

$$\phi'(\langle A \rangle) \leq \phi'(\langle B \rangle).$$

Then

$$\phi'(\langle A \rangle \vee \langle B \rangle) = \phi'(\langle A \rangle) \vee \phi'(\langle B \rangle) = \phi'(\langle B \rangle),$$

so that $\langle A \rangle \vee \langle B \rangle = \langle B \rangle$ and $\langle A \rangle \leq \langle B \rangle$. Thus ϕ' and its inverse are order-preserving, so that ϕ' is a lattice isomorphism. Since ϕ' preserves multiplication, L and L' are isomorphic as multiplicative lattices and thus as Noether lattices.

Now let σ be the restriction of θ to the set of principal elements of RL_n. Then Lemma 1 implies that condition (1) of the theorem is satisfied. Condition (2) of the theorem follows immediately from the definition of θ. To verify condition (4), suppose $\langle 0 \rangle \neq \langle HK \rangle = \langle NK \rangle$. Since L' is a Noether lattice, it follows from condition (2.2) that

$$\langle H \rangle \vee \langle 0 \rangle : \langle K \rangle = \langle N \rangle \vee \langle 0 \rangle : \langle K \rangle.$$

Because $\langle H \rangle \not\leq \langle 0 \rangle : \langle K \rangle$ and $\langle N \rangle \not\leq \langle 0 \rangle : \langle K \rangle$, a minimal representation of $\langle H \rangle \vee \langle 0 \rangle : \langle K \rangle$ has the form

$$\langle H \rangle \vee \langle K_1 \vee \cdots \vee K_t \rangle,$$

and a minimal representation of $\langle N \rangle \vee \langle 0 \rangle : \langle K \rangle$ has the form

$$\langle N \rangle \vee \langle K_1' \vee \cdots \vee K_t' \rangle,$$

with $\langle K_i \rangle \leq \langle 0 \rangle : \langle K \rangle$ and $\langle K_i' \rangle \leq \langle 0 \rangle : \langle K \rangle$. Then since $\langle H \rangle \not\leq \langle 0 \rangle : \langle K \rangle$ and $\langle N \rangle \not\leq \langle 0 \rangle : \langle K \rangle$, it follows by Lemma 1 that $H \; \sigma \; N$. Condition (4) follows from the fact that L' is local.

Now suppose that θ and σ are equivalence relations on RL_n such that conditions (1) to (4) are satisfied. We must verify that RL_n/θ may be regarded as a Noether lattice.

Denote the equivalence class of all elements congruent to A modulo θ by $\langle A \rangle$. Define a partial ordering on $L = RL_n/\theta$ by using equations (3.1) and (3.2). That (3.2) yields a partial ordering follows immediately from the propositions that $\bigvee \langle X \rangle \; \theta \; X$ for all X in RL_n and that $\bigvee \langle X \rangle \leq \bigvee \langle Y \rangle$ whenever $X \leq Y$ in RL_n.

The first of these is clear; to verify the second note that if $X' \; \theta \; X$, then $X' \vee Y \; \theta \; Y$ by condition (1) of the theorem. Then

$$(3.6) \qquad \bigvee \langle X \rangle = \bigvee \{X' \mid X' \; \theta \; X\} \leq \bigvee \{X' \vee Y \mid X' \; \theta \; X\} \leq \bigvee \langle Y \rangle .$$

It follows that (3.3) and (3.4) give the join and meet of any two elements of L relative to the partial ordering defined in (3.2). Clearly, L is complete and is a multiplicative lattice with the multiplication given by (3.5). It is also clear that L satisfies the ascending chain condition. We shall show that L is a Noether lattice by showing that L is distributive and that every element of L is a join of principal elements.

In order to show that L is distributive, we need the relation

$$(3.7) \qquad \bigvee \langle A \vee B \rangle = \left(\bigvee \langle A \rangle\right) \vee \left(\bigvee \langle B \rangle\right).$$

Clearly $\left(\bigvee \langle A \rangle\right) \vee \left(\bigvee \langle B \rangle\right) \leq \bigvee \langle A \vee B \rangle$. Now, assume that

$$C = K_1 \vee \cdots \vee K_r \; \theta \; A \vee B .$$

Let $A = E_1 \vee \cdots \vee E_s$ and $B = H_1 \vee \cdots \vee H_t$. Then, by condition (1), for each K_i there exist integers j(i) and $N_{j(i)}$ such that either

$$K_i \; \theta \; N_{j(i)} \leq E_{j(i)} \quad \text{or} \quad K_i \; \theta \; N_{j(i)} \leq H_{j(i)} .$$

Thus, by (3.6), the elements K_i may be divided into two sets such that the join of the first is less than or equal to $\bigvee \langle A \rangle$, and the join of the second is less than or equal to $\bigvee \langle B \rangle$. Then $C \leq \left(\bigvee \langle A \rangle\right) \vee \left(\bigvee \langle B \rangle\right)$, and therefore the relation (3.7) holds. To see that this implies the distributivity of L, we observe that

$$\begin{aligned} \langle C \rangle \wedge (\langle A \rangle \vee \langle B \rangle) &= \langle C \rangle \wedge \langle A \vee B \rangle \\ &= \left(\bigvee \langle C \rangle\right) \wedge \left[\left(\bigvee \langle A \rangle\right) \vee \left(\bigvee \langle B \rangle\right) \right] \end{aligned}$$

$$= \left[\left(\bigvee\langle C\rangle\right)\wedge\left(\bigvee\langle A\rangle\right)\right]\vee\left[\left(\bigvee\langle C\rangle\right)\wedge\left(\bigvee\langle B\rangle\right)\right]$$
$$= (\langle C\rangle \vee \langle A\rangle)\vee(\langle C\rangle \wedge \langle B\rangle);$$

the second and fourth equality signs are justified by (3.4), and the third by the distributivity of RL_n.

Using (3.2) and (3.5) and the definition of residuation, we obtain the equations

$$\text{(3.8)}\quad \begin{aligned}\langle A\rangle:\langle B\rangle &= \bigvee\{\langle X\rangle \mid \langle X\rangle\langle B\rangle \le \langle A\rangle\} = \bigvee\left\{\langle X\rangle \mid XB \le \bigvee\langle A\rangle\right\}\\ &= \bigvee\left\{\langle X\rangle \mid X \le \left(\bigvee\langle A\rangle\right):B\right\} = \left(\bigvee\langle A\rangle\right):\langle B\rangle.\end{aligned}$$

We shall use (3.8) to show that $\langle E\rangle$ is a principal element in L for each principal element E in RL_n. To show that $\langle E\rangle$ is principal, it is necessary to prove first that

$$\text{(3.9)}\qquad \langle A\rangle \wedge \langle E\rangle = (\langle A\rangle:\langle E\rangle)\langle E\rangle.$$

Note that if $H\ \theta\ K$ in RL_n, then $\langle H\rangle = \langle K\rangle$ and

$$\left\langle\left(\bigvee\langle A\rangle\right):H\right\rangle = \langle A\rangle:\langle H\rangle = \langle A\rangle:\langle K\rangle = \left\langle\left(\bigvee\langle A\rangle\right):K\right\rangle.$$

Multiplying the left side of this expression by $\langle H\rangle$ and the right side by $\langle K\rangle$ and applying equation (2.1) in RL_n, we obtain the equation

$$\text{(3.10)}\qquad \left\langle\left(\bigvee\langle A\rangle\right)\wedge H\right\rangle = \left\langle\left(\bigvee\langle A\rangle\right)\wedge K\right\rangle.$$

Note that since $\bigvee\langle E\rangle\ \theta\ E$, we have the relation $\bigvee\langle E\rangle = E \vee K_1 \vee \cdots \vee K_r$ with $K_i\ \theta\ H_i \le E_j$ by condition (1) of the theorem. Using this and equation (3.8), and applying (3.10) and the fact that $\langle A \vee B\rangle = \langle A\rangle \vee \langle B\rangle$ in the third line below, we obtain the equations

$$\begin{aligned}\langle A\rangle \wedge \langle E\rangle &= \left\langle\left(\bigvee\langle A\rangle\right)\wedge(E\vee K_1\vee\cdots\vee K_r)\right\rangle\\ &= \left\langle\left[\left(\bigvee\langle A\rangle\right)\wedge E\right]\vee\left[\left(\bigvee\langle A\rangle\right)\wedge K_1\right]\vee\cdots\vee\left[\left(\bigvee\langle A\rangle\right)\wedge K_r\right]\right\rangle\\ &= \left\langle\left[\left(\bigvee\langle A\rangle\right)\wedge E\right]\vee\left[\left(\bigvee\langle A\rangle\right)\wedge H_1\right]\vee\cdots\vee\left[\left(\bigvee\langle A\rangle\right)\wedge H_r\right]\right\rangle\\ &= \left\langle\left(\bigvee\langle A\rangle\right)\wedge(E\vee H_1\vee\cdots\vee H_r)\right\rangle\\ &= \left\langle\left(\bigvee\langle A\rangle\right)\wedge E\right\rangle = \left\langle\left[\left(\bigvee\langle A\rangle\right):E\right]E\right\rangle = \langle A:E\rangle\langle E\rangle.\end{aligned}$$

This proves equation (3.9).

The second step in the proof that $\langle E\rangle$ is a principal element is to prove the relation

$$\text{(3.11)}\qquad (\langle B\rangle\langle E\rangle):\langle E\rangle = \langle B\rangle \vee \langle 0\rangle:\langle E\rangle.$$

If $\langle B\rangle\langle E\rangle = \langle 0\rangle$, then $\langle B\rangle \leq \langle 0\rangle : \langle E\rangle$, and (3.11) holds. Assume $\langle B\rangle\langle E\rangle \neq 0$. It is always true that

$$(\langle B\rangle\langle E\rangle) : \langle E\rangle \geq \langle 0\rangle : \langle E\rangle \vee \langle B\rangle;$$

therefore we suppose that K is a principal element in RL_n such that $\langle K\rangle\langle E\rangle \leq \langle B\rangle\langle E\rangle$, with $\langle K\rangle\langle E\rangle \neq 0$. Then $KE \vee BE\ \theta\ BE$, so that (by condition (1) of the theorem) there exists H' such that $H'\ \theta\ KE$ and $H' \leq BE$. But by equation (2.1), $H' = H' \wedge E = HE$ with $H = H' : E$. Therefore $KE\ \theta\ HE$ implies $K\ \theta\ H \leq B$, by condition (3). Therefore $\langle K\rangle \leq \langle B\rangle$. Thus

$$\bigvee \{\langle K\rangle \mid \langle K\rangle\langle E\rangle \leq \langle B\rangle\langle E\rangle \text{ and } \langle K\rangle\langle E\rangle \neq \langle 0\rangle\} \leq \langle B\rangle.$$

Hence equation (3.11) holds. Thus, by Theorem 1, E is a principal element, so that each element of L is a join of principal elements. Therefore L is a distributive Noether lattice. L is local, for if $\langle A\rangle \vee \langle B\rangle = \langle I\rangle$, then $A \vee B = I$ by condition (4), so that $A = I$ or $B = I$. This proves the theorem.

4. EXAMPLES OF DISTRIBUTIVE LOCAL NOETHER LATTICES

We can use Theorem 2 to construct interesting examples of Noether lattices. For example, the lattice L obtained by identifying $(x_1)^2$ and $(x_2)^2$ in RL_2 is drawn schematically in Figure 1. The dots indicate that the pattern above them is to be continued.

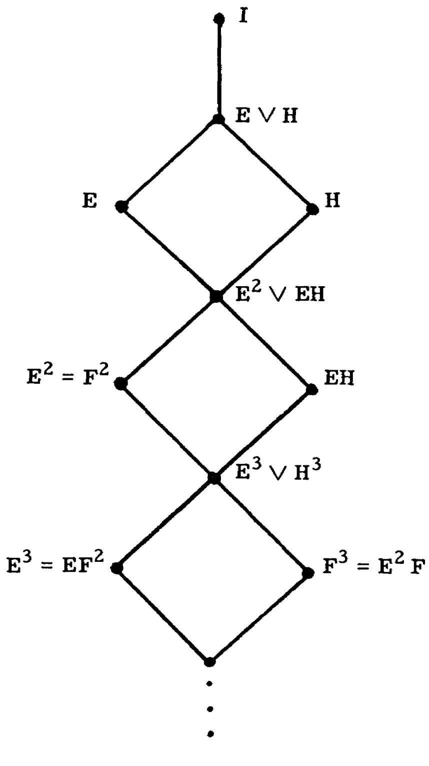

Figure 1.

The lattice obtained from RL_2 by identifying $(x_1)^2$ and $(x_2)^2$ as described in Theorem 2.

To verify that L is the lattice shown in the figure, let $\langle (x_1) \rangle = E$ and $\langle (x_2) \rangle = H$. Note first that each principal element of L has a factorization of the form E^i or $E^i H$. Thus every element X of L has the form $X = E^i \vee E^j H$, where i denotes the smallest integer such that E^i is contained in X, and where j denotes the smallest integer such that $E^j H \leq X$. If $j \geq i$, then $X = E^i$. If $j < i$, then

$$E^i \vee E^j H = E^j (E^{i-j} \vee H) = \begin{cases} E^j (E \vee H) & \text{if } i - j = 1, \\ E^j H & \text{if } i - j > 1. \end{cases}$$

Therefore each element of L has the form E^i, $E^j H$, or $E^k (E \vee H)$, and Figure 1 is the correct diagram for L.

Another interesting example that may be obtained from RL_2 is

$$L' = RL_2 / [(x_1) \vee (x_2)]^2 .$$

Although the quotient lattice L/D is defined differently in [2] from the way L/θ is defined in this paper, it is easy to see that L' is the lattice obtained from RL_2 by identifying $(x_1)^2$, $(x_1)(x_2)$, and $(x_2)^2$ with 0 as described in Theorem 3.1. The diagram for L' may thus be obtained from the diagram for L by taking $E^2 \vee EH$ as the zero element in Figure 1. L' arises naturally as a quotient sublattice of any distributive local Noether lattice that is not a chain. This is a crucial point in the proof of the following theorem.

THEOREM 3. *If* L_R *is the lattice of ideals of a local ring* R, *then* L_R *is distributive if and only if* L_R *is a chain.*

Proof. If the maximal ideal of R is principal, L_R is a chain. Suppose the maximal ideal of R has a minimal representation of the form $E_1 \vee \cdots \vee E_K$, where each E_i is a principal ideal. Let

$$A = (E_1 \vee E_2)^2 \vee E_3 \vee \cdots \vee E_K ,$$

and let X' denote $X \vee A$. Then one can show easily that $E_1' \vee E_2'$, E_1', E_2', and 0' are distinct elements of L_R/A. For example, if $E_1' = E_2'$, then

$$\begin{aligned} I = (E_1 \vee A) : E_1 &= (E_2 \vee A) : E_1 \\ &= (E_1^2 \vee E_2 \vee \cdots \vee E_K) : E_1 = E_1 \vee (E_2 \vee \cdots \vee E_K) : E_1 . \end{aligned}$$

Since L_R is local, $(E_2 \vee \cdots \vee E_K) : E_1 = I$. This is impossible, since it implies that $E_1 \leq E_2 \vee \cdots \vee E_K$.

Now suppose that $X' \neq I$ is an element of L_R/A. Then

$$\begin{aligned} X' &= X' \wedge (E_1' \vee E_2') = (X' \wedge E_1') \vee (X_2' \wedge E_2') \\ &= (X' : E_1') E_1' \vee (X' : E_2') : E_2' . \end{aligned}$$

But if $X' : E_j' = I$, then $(X' : E_j') E_j' = 0$. Thus I', $E_1' \vee E_2'$, E_1', E_2', and 0' are the only elements of L_R/A. The lattice L_R/A is the lattice of ideals of the ring R/A. Since E_1' and E_2' are join-irreducible, they must be principal ideals. Let $E_1' = (x)$ and $E_2' = (y)$. Since $(x) \neq (y)$, it follows that $(x + y) \neq 0$. Also $(x + y) \neq (x)$, for if

$x + y = rx$, then y is in (x), which is impossible. If $(x + y) = (x, y)$, then (x, y) is principal and Corollary 2.2 of [1] implies that $(x, y) = (x)$ or $(x, y) = (y)$. Again this is impossible, so that L_R/A cannot be the lattice of all ideals of any ring. Thus the maximal ideal of R is principal and L_R is a chain.

Theorem 3 is a special case of a theorem that will appear in a paper of E. W. Johnson and J. P. Lediaev [3]. The proof given here is different from theirs.

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