PRODUCTS OF SELF-ADJOINT OPERATORS

Heydar Radjavi and James P. Williams

Introduction. The purpose of this paper is to present some partial results concerning the problem of characterizing the bounded linear operators T on a Hilbert space H that admit a factorization as a product of two self-adjoint operators. We conjecture that an invertible operator T has this property if and only if T is similar to its adjoint. The main results are (a) a proof of the conjecture under the restriction that $\dim H < \infty$, and (b) a characterization of the operators that are unitarily equivalent to their adjoints. We also establish other sufficient conditions under which the conjecture is true.

1. We begin by considering the finite-dimensional case. Theorem 1 gives a reasonably good characterization of the product of two self-adjoint operators.

THEOREM 1. *If* H *is a finite-dimensional Hilbert space, then the following are equivalent conditions for an operator* T *on* H.

(1) T *is a product of two self-adjoint operators.*

(2) T *is a product of two self-adjoint operators, one of which is invertible.*

(3) *There exists an invertible self-adjoint operator* A *such that* TA *is self-adjoint.*

(4) *There exists an invertible self-adjoint operator* A *such that* $A^{-1}TA = T^*$.

(5) *There exists a basis of* H *with respect to which the matrix of* T *is real.*

(6) T *is similar to* T^*.

Proof. Carlson [1] proved the equivalence of the first five conditions. For the sake of completeness, we include here a substantial simplification of his arguments.

The implications (2) ⇒ (4) ⇒ (3) ⇒ (2) ⇒ (1) are clear, and therefore it suffices to prove (1) ⇒ (2) and the chain (5) ⇒ (6) ⇒ (2) ⇒ (6) ⇒ (5).

(5) ⇒ (6). Suppose that T has real matrix (a_{ij}) relative to the basis $\{e_i\}$. Choose an orthonormal basis $\{f_i\}$, and define an invertible operator S so that $Sf_i = e_i$. Then $S^{-1}TS$ has matrix (a_{ij}) relative to the orthonormal basis $\{f_i\}$. Since any matrix is similar to its transpose, it follows that $S^{-1}TS$ is similar to $(S^{-1}TS)^t = (S^{-1}TS)^*$. This implies that T is similar to T^*.

(6) ⇒ (2). Assume that $TS = ST^*$ for some invertible operator S. Taking adjoints, one sees easily that

$$T(e^{i\theta} S + e^{-i\theta} S^*) = (e^{i\theta} S + e^{-i\theta} S^*)T^*$$

for each real θ. Now the operator

$$A_\theta = e^{i\theta} S + e^{-i\theta} S^* = (SS^{*-1} + e^{-2i\theta})e^{i\theta} S^*$$

Received May 16, 1968.

is invertible if we choose θ so that $-e^{-2i\theta}$ does not belong to the spectrum of SS^{*-1}. (This choice is possible because $\sigma(SS^{*-1})$ is a finite set.) Hence, for suitable θ,

$$T = (TA_\theta)A_\theta^{-1},$$

so that T satisfies condition (2).

The implication (2) ⇒ (6) is trivial: If $T = AB$, where A and B are self-adjoint and (say) A is invertible, then $TA = AT^*$, so that T is similar to T^*.

(6) ⇒ (5). It suffices to show that if the matrix A is similar to its adjoint, then A is also similar to a real matrix. Now, since any matrix is similar to its transpose, the hypothesis insures that A is similar to its complex conjugate $\bar{A}$. This means that the Jordan canonical form of A has the form

$$\mathrm{diag}\,\{A_1, \cdots, A_p, \bar{A}_1, \cdots, \bar{A}_p, R_1, \cdots, R_q\},$$

where the matrices R_i are real and each A_i has the form

$$\begin{pmatrix} \lambda & & & & \\ 1 & \cdot & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & 1 & \lambda \end{pmatrix}.$$

Hence it suffices to show that a matrix of the form $\mathrm{diag}\,\{A, \bar{A}\}$ is similar to a real matrix. Such a matrix is actually unitarily equivalent to a real matrix. In fact, if A and B are $n \times n$ matrices, and if I is the $n \times n$ identity, then the matrix

$$\begin{pmatrix} \frac{1}{\sqrt{2}} I & \frac{i}{\sqrt{2}} I \\ \frac{i}{\sqrt{2}} I & \frac{1}{\sqrt{2}} I \end{pmatrix}^{-1} \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} I & \frac{i}{\sqrt{2}} I \\ \frac{i}{\sqrt{2}} I & \frac{1}{\sqrt{2}} I \end{pmatrix}$$

is real.

It remains to prove that (1) ⇒ (2). Suppose then that $T = AB$, where A and B are self-adjoint. The identity $S^{-1}(AB)S = (S^{-1}AS^{*-1})(S^*BS)$ shows that the set of products AB of self-adjoint operators is invariant under similarity. Hence, replacing T by a similarity of T if necessary, we may assume that T is the direct sum of a nilpotent N and an invertible R.

Now $TA = AT^*$ implies $T^nA = AT^{*n}$ for every natural number n. If A is partitioned suitably, this last equation may be expressed as

$$\begin{pmatrix} N^n & 0 \\ 0 & R^n \end{pmatrix} \begin{pmatrix} A_1 & C \\ C^* & A_2 \end{pmatrix} = \begin{pmatrix} A_1 & C \\ C^* & A_2 \end{pmatrix} \begin{pmatrix} N^{*n} & 0 \\ 0 & R^{*n} \end{pmatrix}.$$

It follows that $R^nC^* = C^*N^{*n}$ for each n, and this implies $C = 0$.

Let $\begin{pmatrix} B_1 & D \\ D^* & B_2 \end{pmatrix}$ be the corresponding partition of B. Then the equation $T = AB$ gives the relation $R = A_2 B_2$. Since R is invertible, the self-adjoint operators A_2 and B_2 are also invertible. Hence $A_2^{-1} R A_2 = R^*$.

Finally, N is nilpotent and is therefore similar to its adjoint, so that there exists an invertible A_3 with $A_3^{-1} N A_3 = N^*$. The operator $S = \operatorname{diag}\{A_3, A_2\}$ then implements the similarity between T and T^*. This completes the proof of Theorem 1.

It is reasonable to ask whether the existence of an *orthonormal* basis in (5) is equivalent to T being *unitarily* equivalent to its adjoint in (6). However, if

$$T_1 = \begin{pmatrix} 0 & 0 & 1 & i \\ 0 & 1 & 1 & 1 \\ -1 & -1 & 2 & 0 \\ i & -1 & 0 & 3 \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 1 \end{pmatrix},$$

then T_1 is unitarily equivalent to its adjoint but does not have a real matrix in any basis; even though T_2 has a real matrix, the similarity between T_2 and its adjoint cannot be implemented by a unitary operator.

2. In the remainder of this paper, we shall study the conditions in Theorem 1 in the case where H is an infinite-dimensional Hilbert space. In the present section, we provide some motivation for what follows.

We begin by listing some known properties of products of self-adjoint operators.

PROPOSITION. *If* A *and* B *be self-adjoint, then*

(i) *the spectrum of* AB *is symmetric with respect to the real axis, and the residual spectrum of* AB *contains no nonzero real number;*

(ii) AB *is invertible if and only if both* A *and* B *are invertible;*

(iii) *in case* A *and* B *are projections,* AB *is unitarily equivalent to its adjoint and has nonnegative spectrum;*

(iv) AB *and* BA *need not be similar.*

Proof. It is well known that if X and Y are operators, then the nonzero points of $\sigma(XY)$ and $\sigma(YX)$ are identical. Moreover, it is easy to see that if the product AB of two self-adjoint operators is invertible, then both A and B are invertible. Thus

$$\sigma(AB) = \sigma(BA) = \sigma((AB)^*) = \sigma(AB)^-,$$

where the bar denotes complex conjugation. This proves (ii) and the symmetry assertion in (i).

It is clear that if λ is real and nonzero, then $AB - \lambda = (BA - \lambda)^*$ is invertible if and only if it is bounded below, because

$$B \operatorname{null}(AB - \lambda) = \operatorname{null}(BA - \lambda).$$

This completes the proof of (i).

Dixmier [4] proved that a product of two projections is unitarily equivalent to its adjoint. The nonnegativity of the spectrum in (iii) is obtained from the well-known relation

$$\sigma(AB) = \sigma(A^{1/2}A^{1/2}B) = \sigma(A^{1/2}BA^{1/2}) \pm \{0\}.$$

Finally, to prove (iv) we observe that the null spaces of two similar operators have the same dimension. Hence it suffices to exhibit self-adjoint operators A and B such that AB is one-to-one, but BA is not. This is easy to do. For example, let A be a positive operator with dense range, let P be a projection whose range is disjoint from the range of A, say $Px = (x, y)y$, where y does not lie in the range of A, and put $B = P^{\perp}$.

Remark. In view of the preceding example, it is reasonable to ask whether AB and BA are similar if their null spaces have the same dimension. We do not know the answer.

Consider now the conditions of Theorem 1. The implications

$$(6) \Leftarrow (4) \Leftrightarrow (3) \Leftrightarrow (2) \Rightarrow (1)$$

are of course purely formal, and hence they remain valid in the infinite-dimensional case. On the other hand, we have just shown that (1) and (2) are not equivalent, and it is easy to see that (5) and (6) are also not equivalent. (For example, the unilateral shift defined on an orthonormal basis $\{e_n\}_0^\infty$ by $Ue_n = e_{n+1}$ has real matrix, but it is not similar to its adjoint, because U^* has a null space and U does not.) Thus the only possible nontrivial relation is (6) $\Leftrightarrow$ (4). This may be equivalently stated as follows.

Conjecture. An invertible operator T is a product of two self-adjoint operators if and only if T is similar to T^*.

For the sake of brevity, we denote by $\mathscr{C}_0$ the set of all invertible products of self-adjoint operators A and B, and by $\mathscr{C}$ the set of invertible operators that are similar to their adjoints. We have the inclusion relation $\mathscr{C}_0 \subseteq \mathscr{C}$, and the conjecture asserts that its reverse is also valid.

We shall use the invariance of the classes $\mathscr{C}_0$ and $\mathscr{C}$ under similarity transformations $T \to S^{-1}TS$. We also notice that $\mathscr{C}$ is strictly larger than the class of operators that are similar to self-adjoints. (The bilateral shift is a counterexample).

There is no relation between the operators in $\mathscr{C}$ and the *symmetrizable* [5], [8] or *quasi-Hermitian* [3] operators. Thus the adjoint of the operator $AP^{\perp}$ constructed in the proof of (v) of our proposition is quasi-Hermitian but does not belong to $\mathscr{C}$. In the other direction, $\mathscr{C}$ contains nilpotents, but there exist no nontrivial quasi-nilpotent, quasi-Hermitian operators [3].

3. In the proof of Theorem 1, we observed that the equation $TS = ST^*$ implies

$$T[\Re(e^{i\theta} S)] = [\Re(e^{i\theta} S)]T^*$$

for all θ. Hence, if we can choose θ so that the operator in brackets is invertible, then $T \in \mathscr{C}_0$. This choice is possible, for example, if the imaginary part of S is compact. In fact, this assumption implies that SS^{*-1} is the sum of the identity and a compact operator whose spectrum cannot contain $-e^{-2i\theta}$ for all θ.

If S is unitary, an appropriate choice of θ may not be possible, but our technique yields the following result [12]. (Here $W(S) = \{(Sx, x): \|x\| = 1\}$ is the numerical range of S, and the bar denotes closure.)

THEOREM 2. *The following are equivalent conditions on an operator* T:

(1) T *is similar to a self-adjoint operator.*

(2) $T = PA$, *where* P *is positive and invertible and* A *is self-adjoint.*

(3) $S^{-1}TS = T^*$ *and* $0 \in W(S)^-$.

Theorem 2 gives a sufficient condition for an operator $T \in \mathscr{C}$ to belong to $\mathscr{C}_0$, namely, that the similarity between T and T^* can be implemented by an operator S that is strongly invertible in the sense of condition (3). If S is unitary, then this condition amounts to the assumption that the spectrum of S is contained in an arc of the unit circle of length less than π. The next result deals with the situation in which this restriction on the spectrum is not satisfied.

THEOREM 3. T *is unitarily equivalent to its adjoint if and only if* T *is the product of a symmetry and a self-adjoint operator.*

Proof. If $T = JA$, where $J = J^* = J^{-1}$ is a symmetry and A is self-adjoint, then $JTJ = AJ = T^*$, so that T is unitarily equivalent to its adjoint.

Conversely, suppose $TU = UT^*$, where U is unitary. Then T commutes with U^2. Let $U^2 = \int e^{i\theta}\, dE_\theta$ be the spectral representation of U^2. If $V = \int e^{i\theta/2}\, dE_\theta$, then V is a unitary operator, $V^2 = U^2$, and V commutes with every operator that commutes with U^2. It follows that V commutes with U and T, and therefore $J = V^{-1}U$ is a symmetry and $TJ = JT^*$. Hence $T = J(TJ)$ is the product of a symmetry and a self-adjoint operator.

COROLLARY 1. *Each normal operator in* $\mathscr{C}$ *belongs to* $\mathscr{C}_0$.

Proof. Two normal operators that are similar are also unitarily equivalent, by the Putnam-Fuglede Theorem [6].

COROLLARY 2. *If* $T \in \mathscr{C}$ *and* T *is similar to a normal operator, then* $T \in \mathscr{C}_0$.

Proof. $\mathscr{C}_0$ is invariant under similarity transformations.

COROLLARY 3. *Assume* $S^{-1}TS = T^*$. *If* S *is congruent to a normal operator* ($S = RNR^*$, N *normal*), *then* $T \in \mathscr{C}_0$.

Proof. We have the relation $T_0 N = NT_0^*$, where $T_0 = R^{-1}TR$ is similar to T. Hence it suffices to show that $T_0 \in \mathscr{C}_0$. Now, if $N = UP$ is the polar decomposition of N, then commutativity of U and P implies that

$$(P^{-1/2}T_0 P^{1/2})U = U(P^{-1/2}T_0 P^{1/2})^* .$$

Since U is unitary, it follows from Theorem 3 that $P^{-1/2}T_0 P^{1/2}$ belongs to $\mathscr{C}_0$. This implies that $T_0 \in \mathscr{C}_0$.

COROLLARY 4. *Assume* $TS = ST^*$, *where* S *is invertible. Then the following conditions on* S *are equivalent:*

$$\text{(i) } ST = T^*S, \qquad \text{(ii) } S^2T = TS^2, \qquad \text{(iii) } S^*ST = TS^*S.$$

Moreover, if any of these additional conditions is satisfied, then $T \in \mathscr{C}_0$.

Proof. If $TS = ST^*$, the equivalence of (i), (ii), (iii) is trivial.

Suppose now that the additional conditions hold, and let $S = VP$ be the polar decomposition of S. Since $TS = ST^*$, the operator $S^* S^{-1}$ commutes with T. It follows that $P^2 = S^* S = (S^* S^{-1})S^2$ also commutes with T, and hence that $PT = TP$. Therefore $PT^* = T^* P$, and it follows from the equation $TS = ST^*$ that $TV = VT^*$. Hence, by Theorem 3, T is the product of a symmetry and a self-adjoint operator.

Theorem 3 contains other facts of interest. For example, the following result of C. Davis [2] is an immediate consequence.

THEOREM 4. *A unitary operator* U *is similar to its inverse if and only if* U *is the product of two symmetries.*

Applying Theorem 3 to the operator iA, where A is self-adjoint, we get a result about self-commutators, that is, operators of the form $T^* T - TT^*$:

COROLLARY 5. *A self-adjoint operator* A *is similar to its negative if and only if* $A = i(JB - BJ)$, *where* J *is a symmetry and* B *is self-adjoint.*

Concerning Corollary 3 of Theorem 3, we do not know which invertible operators S are congruent to a normal operator. However, it is easy to see that the following are equivalent:

(i) $S = R^* NR$ (N normal, R invertible),

(ii) $S = PNP$ (P positive and invertible, N normal),

(iii) $S = PUP$ (P positive and invertible, U unitary).

Moreover, each of these conditions implies that SS^{*-1} is similar to a unitary operator. (This allows us to exhibit 2×2 matrices S that are not congruent to a normal operator.)

The class of operators that are unitarily equivalent to their adjoints is not as well-behaved as one might expect. For example, the Bishop operator defined on $L^2(0, 1)$ by

$$(Bf)(x) = x \cdot f(x + \alpha)$$

(where α is irrational and the addition is modulo 1) is a prime candidate for an example of an operator with no nontrivial invariant subspaces. Yet iB is unitarily equivalent to its adjoint. (The relevant symmetry is $(Jf)(x) = f(\alpha - x)$.)

We conclude this section with another sufficient condition for an operator in $\mathscr{C}$ to belong to $\mathscr{C}_0$.

THEOREM 5. *If* T *has no real spectrum and if* $T \in \mathscr{C}$, *then* $T \in \mathscr{C}_0$.

Proof. Let σ_1 and σ_2 be the parts of $\sigma(A)$ lying in the upper and lower half-planes, respectively. Since T is similar to T^*, σ_2 is the complex conjugate of σ_1. By the decomposition theorem of Riesz [9, p. 421], the Hilbert space H is the direct sum of two linearly independent subspaces H_1 and H_2, where H_i is invariant under T and the spectrum of the restriction of T to H_i is σ_i $(i = 1, 2)$.

Writing every vector x in H as $x_1 + x_2$ with $x_i \in H_i$, we consider the new norm $|x| = \|x_1\| + \|x_2\|$ on H and observe that the identity transformation from $(H, |\cdot|)$ onto $(H, \|\cdot\|)$ is one-to-one and bounded. Hence its inverse is bounded, by the closed-graph theorem, so that there exists an $m > 0$ such that

$$m(\|x_1\| + \|x_2\|) \leq \|x_1 + x_2\|$$

for every x. It follows that the operator R defined by the linear extension of

$$Rx = \begin{cases} x & \text{for } x \in H_1, \\ x_2 & \text{for } x \in H_1^{\perp} \end{cases}$$

is invertible, and $R^{-1}TR$ is a similarity of T leaving H_1 and $H_1^{\perp}$ invariant. Hence we can assume, without loss of generality, that $T = \operatorname{diag}\{T_1, T_2\}$, with $\sigma(T_i) = \sigma_i$.

By hypothesis, $TS = ST^*$, for some invertible S. If $S = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}$, then $T_1 S_1 = S_1 T_1^*$ and $T_2 S_4 = S_4 T_2^*$. Since $\sigma(T_i) \cap \sigma(T_i^*)$ is empty $(i = 1, 2)$, it follows from a result of Rosenblum [10] that $S_1 = S_4 = 0$. Hence S_2 and S_3 are invertible, and $T_2 = S_3 T_1^* S_3^{-1}$. Hence

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & S_3 T_1^* S_3^{-1} \end{pmatrix} = \begin{pmatrix} 0 & S_3^* \\ S_3 & 0 \end{pmatrix} \begin{pmatrix} 0 & T_1^* S_3^{-1} \\ S_3^{*-1} T_1 & 0 \end{pmatrix};$$

in other words, $T \in \mathscr{C}_0$.

4. The next result gives some information about the class $\mathscr{C}_0$ of invertible products of two self-adjoint operations.

THEOREM 6. *The following conditions on* T *are equivalent.*

(1) $T \in \mathscr{C}_0$.

(2) *There exist a symmetry* J *and a positive invertible* P *such that*

$$(PJP)^{-1} T (PJP) = T^*.$$

(3) *There exist a symmetry* J *and a positive invertible* P *such that*

$$J(P^{-1} T P)J = (P^{-1} T P)^*.$$

(4) *There exist an invertible* R *and a normal* N *such that*

$$(R^* NR)^{-1} T (R^* NR) = T^*.$$

(5) *Some similarity of* T *is unitarily equivalent to its adjoint.*

Proof. The implications (4) $\Leftarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (5) are clear. Theorem 3 and the invariance of $\mathscr{C}_0$ under similarity yield the result (5) $\Rightarrow$ (1).

The implication (4) $\Rightarrow$ (1) is Corollary 3 of Theorem 3.

It remains to show that (1) $\Rightarrow$ (2). For this, write $T = AB$, where A and B are self-adjoint and A is invertible, and let $A = JP$ be the polar decomposition of A. Here P is positive, J is a symmetry, and J and P commute. Then

$$(P^{1/2} J P^{1/2})^{-1} T (P^{1/2} J P^{1/2}) = T^*,$$

so that condition (2) holds for T.

The preceding theorem permits a reformulation of the conjecture: If T is similar to its adjoint, then some similarity of T is unitarily equivalent to its adjoint.

5. There is a connection between the operators in $\mathscr{C}$ and self-commutators. A self-adjoint operator is a self-commutator if and only if it has the form i(AB - BA), where A and B are self-adjoint.

It is clear that if T = AB is a product of two self-adjoint operators, then the imaginary part of T is a self-commutator. Consequently, if the conjecture is true, then the same must be true for each $T \in \mathscr{C}$. The next result is therefore of interest.

THEOREM 7. *If* T *is similar to* T^*, *then* $\Im T$ *is a self-commutator.*

Proof. Without loss of generality, we may suppose that H is separable. In this case, self-commutators may be characterized [7] as the self-adjoint operators that are not of the form $K \pm M$, where K is compact and $M \geq \delta > 0$.

If the operator T satisfies the condition $\Im T \geq \delta > 0$, then the numerical range of T lies in the half-plane $\{\Im z \geq \delta\}$. Since the closure of the numerical range contains the spectrum, it follows that $\sigma(T)$ cannot be symmetric with respect to the real axis. Hence, if $S^{-1} T S = T^*$ for some invertible S, then neither of the conditions $\Im T \geq \delta$ and $\Im T \leq -\delta$ can hold for $\delta > 0$.

Now one can define the numerical range $W_0(\hat{T})$ of an element $\hat{T}$ of the quotient algebra $\mathscr{B}(H)/\mathscr{K}$, $\mathscr{K}$ being the ideal of compact operators [11]. Here $W_0(\hat{T})$ is convex and contains $\sigma(\hat{T})$, and therefore the above argument, carried out in the algebra $\mathscr{B}(H)/\mathscr{K}$, yields the conclusion that $\Im T$ cannot be of the form $K \pm M$, where K is compact and $M \geq \delta > 0$. This completes the proof.

We conclude with three observations that are relevant to the problem of giving an intrinsic characterization of the product of two self-adjoint operators.

THEOREM 8 (T. Crimmins). T *is a product of two projections if and only if* $TT^* T = T^2$.

Proof. The necessity is trivial. To prove the sufficiency, let P be the projection onto the closure of the range of T, and let Q be the projection onto the orthogonal complement of the null space of T. Then T = PQ.

THEOREM 9. *In order that* T = PA, *where* P *is a projection and* A *is self-adjoint, it is necessary and sufficient that* $T^{*2} T = T^* T^2$.

Proof. Let P be defined as in the proof of Theorem 8, and note that

$$A = PTP + TP^{\perp} + P^{\perp} T^*$$

is self-adjoint.

THEOREM 10 (J. Stampfli). $T^* T T^* = T T^* T$ *if and only if* T *is self-adjoint.*

Proof. $(T^* T)^3 = (T^* T T^*)(T T^* T) = (T T^* T)(T^* T T^*) = (T T^*)^3$.

REFERENCES

1. D. H. Carlson, *On real eigenvalues of complex matrices.* Pacific J. Math. 15 (1965), 1119-1129.
2. C. Davis, *Separation of two linear subspaces.* Acta. Sci. Math. (Szeged) 19 (1958), 172-187.
3. J. Dieudonné, *Quasi-hermitian operators.* Proc. Internat. Sympos. Linear Spaces (Jerusalem, 1960), pp. 115-123. Jerusalem Academic Press, Jerusalem; Pergamon, Oxford, 1961.
4. J. Dixmier, *Position relative de deux variétés linéaires fermées dans un espace de Hilbert.* Revue Sci. 86 (1948), 387-399.
5. P. D. Lax, *Symmetrizable linear transformations.* Comm. Pure Appl. Math. 7 (1954), 633-647.
6. C. R. Putnam, *On normal operators in Hilbert space.* Amer. J. Math. 73 (1951), 357-362.
7. H. Radjavi, *Structure of* $A^*A - AA^*$. J. Math. Mech. 16 (1966), 19-26.
8. W. T. Reid, *Symmetrizable completely continuous linear transformations in Hilbert space.* Duke Math. J. 18 (1951), 41-56.
9. F. Riesz and B. Sz.-Nagy, *Functional analysis.* Ungar, New York, 1955.
10. M. Rosenblum, *On the operator equation* $BX - XA = Q$. Duke Math. J. 23 (1956), 263-269.
11. J. G. Stampfli and J. P. Williams, *Growth conditions and the numerical range in a Banach algebra* (to appear).
12. J. P. Williams, *Operators similar to their adjoints.* Proc. Amer. Math. Soc. (to appear).

Pahlavi University, Shiraz, and University of Toronto
Indiana University, Bloomington, Indiana 47401