ON THE NORMAL TYPE OF A FINITE COMPLEX

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Introduction. We define the *normal type* $\eta(K)$ of a finite complex K as follows. Let M be a compact smooth manifold having the homotopy type of K, and let F_M denote the homotopy fiber of the inclusion $\partial M \to M$. Then $\eta(K)$ is defined to be the S-type of F_M.

In this paper we show that $\eta(K)$ does not depend on the choice of M, but only on the homotopy type of K. We also show that $\eta(K \times L) = \eta(K) * \eta(L)$, so that η is a homomorphism of semigroups.

1. Let M^m be a compact, smooth, 1-connected manifold with nonempty, connected boundary. Replacing the inclusion $i: \partial M \to M$ by a fibering in the usual way, we obtain the fiber

$$F_M = \{\alpha \in M^I |\ \alpha(0) = * \text{ and } \alpha(1) \in \partial M\}.$$

The following is our main result; the proof will be given in Section 3.

THEOREM 1. *The* S*-type of* F_M *depends only on the homotopy type of* M.

This is in contrast with the situation for the cofiber $M/\partial M$. For example, let M and M' be the total spaces of k-disk bundles ξ and ξ' over a closed manifold V^{m-k} such that $J(\xi) = 0$, $J(\xi') \neq 0$. Then $M/\partial M$ and $M'/\partial M'$ are the corresponding Thom complexes, and $M/\partial M$ is S-coreducible, while $M'/\partial M'$ is not. Observe that $M \sim V \sim M'$ and $F_M \sim S^{k-1} \sim F_{M'}$. This example suggests the following theorem [2].

THEOREM 2. *Let* M *be* 1*-connected. Then* F_M *has the* S*-type of a sphere* S^{k-1} *if and only if* M *is a Poincaré complex of formal dimension* m - k.

It follows from Theorem 2 that the elements of ker η are precisely the Poincaré complexes. Our next theorem says that F is multiplicative.

THEOREM 3. $F_{M \times N}$ *has the homotopy type of* $F_M * F_N$.

COROLLARY. $F_{I \times M} \sim S^0 * F_M \sim S^1 \wedge F_M$.

THEOREM 4. *If* $A \times B$ *is a* 1*-connected Poincaré complex, then the same is true of* A *and* B.

Proof. By Theorems 1 and 2, it is enough to take M and N of the homotopy types of A and B, and to show that F_M and F_N are homology spheres. We see from Theorems 2 and 3 that $F_M * F_N$ is a homology sphere, and the result now follows from the Künneth formula.

2. Despite the noninvariance of $M/\partial M$, the Poincaré duality theorem states that $H_i(M/\partial M)$ is isomorphic to $H^{m-i}(M)$. We do not have a good formula for $H_*(F_M)$. Using the Serre spectral sequence for $F_M \to \partial M \to M$, we obtain the following result.

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THEOREM 5. *Let* M *be* 1-*connected; then either* F_M *is a sphere, or* $H_*(F_M)$ *is not of finite dimension.*

Proof. For simplicity, we prove this for rational coefficients; the proof for Z-coefficients is similar. Let $H_{k-1}(F_M, Q)$ be the first nonzero rational homology group. Then we have isomorphisms

$$H_{k-1}(F_M, Q) \approx H_k(M/\partial M, Q) \approx H^{m-k}(M, Q) \approx H_{m-k}(M, Q),$$

and $H_i(M, Q) = 0$ for $i > m - k$. Then, in the spectral sequence,

$$E^2_{m-k,k-1} = H_{m-k}(M, Q) \otimes H_{k-1}(F_M, Q) \neq 0.$$

If $\ell > k - 1$, then $m - k + \ell > m - 1$, and $E^\infty_{m-k,\ell} = 0$.

If $H_\ell(F_M, Q) \neq 0$ $(\ell > k - 1)$, then some outgoing differential d^r must be nonzero at $(m - k, \ell)$, since the incoming differentials there are all zero. Thus the condition that $H_\ell(M, Q) \neq 0$ $(\ell > k - 1)$ implies that $H_q(F_M, Q) \neq 0$ for some $q > \ell$, and thus that $H_*(F_M, Q)$ is not of finite dimension. If $H_\ell(F_M, Q) = 0$ for all $\ell > k - 1$, then

$$E^\infty_{m-k,k-1} = E^2_{m-k,k-1} = H_{m-k}(M, Q) \otimes H_{k-1}(F_M, Q);$$

but $H_{m-1}(\partial M, Q) = Q$, so that $H_{k-1}(F_M, Q)$ has rank 1. ■

3. In order to prove Theorem 1, we first establish a special case:

THEOREM 6. *Let* M^m *be a compact, smooth,* 1-*connected manifold with non-empty boundary, and let* $D^k \to N \xrightarrow{\pi} M$ *be a* k-*disk bundle over* M. *Let* $\sigma\colon M \to N$ *be the zero section, and let* $E \xrightarrow{p} M$ *be the associated sphere bundle. Then there is a commutative diagram*

$$\begin{array}{ccc} F_M * S^{k-1} & \xrightarrow{f} & F_N \\ \downarrow & & \downarrow \\ \overline{\partial M} \oplus E & \xrightarrow{\phi} & \overline{\partial N} \\ \downarrow & & \downarrow \\ M & \xrightarrow{\sigma} & N \end{array}$$

in which ϕ *and* f *are equivalences.*

Proof. It is enough to define ϕ and f, to verify commutativity, and to show that f is a homotopy equivalence. Recall that $\overline{\partial M} = \{\alpha \in M^I \mid \alpha(1) \in \partial M\}$, where $\overline{\partial M} \to M$ denotes evaluation at $0 \in I$. The space $\overline{\partial M} \oplus E$ can be defined as a quotient space of the set of triples (α, θ, x) with $\alpha \in \overline{\partial M}$, $0 < \theta < \pi/2$, $x \in E$, and $\alpha(0) = p(x)$. Choose a Hurewicz lifting function L for $N \xrightarrow{\pi} M$. Now, given a triple (α, θ, x) representing an element of $\overline{\partial M} \oplus E$, define $\phi(\alpha, \theta, x)$ as follows. First, x determines a line segment in N from x to $p(x) = \alpha(0)$. Translate this segment L_x along α to obtain a square in N, that is, a map

$$\Gamma_{\alpha,x}\colon I \times I \to N \quad \text{with} \quad \Gamma(t, 0) = L_x(t) \quad \text{and} \quad \Gamma(0, s) = \alpha(s).$$

Now let $\hat{\theta}$ be the line from (0, 0) to $\partial[I \times I]$ with angle θ, so that $\hat{\theta}: I \to I \times I$. Define $\phi(\alpha, \theta, x) \in \overline{\partial N}$ by $\phi(\alpha, \theta, x) = \Gamma_{\alpha,x} \cdot \hat{\theta}$. It is easy to verify that ϕ takes values in $\overline{\partial N}$, is well-defined on $\overline{\partial M} \oplus E$, and covers σ. Thus ϕ induces

$$f: F_M * S^{k-1} \to F_N$$

by restriction. To see that f is a homotopy equivalence, write $\partial N = E \cup \pi^{-1}\partial M$; then $F_N = A \cup B$, where

$$A = \{\alpha \in F_N \mid \alpha(1) \in E\} \quad \text{and} \quad B = \{\alpha \in F_N \mid \alpha(1) \in \pi^{-1}\partial M\} .$$

One can show that (F_N, A, B) is a proper triad. Similarly, let

$$X = \{(\alpha, \theta, x) \mid \theta \geq \pi/4\} \quad \text{and} \quad Y = \{(\alpha, \theta, x) \mid \theta \leq \pi/4\} .$$

Then f is a map of triads:

$$(F_M * S^{k-1}, X, Y) \to (F_N, A, B).$$

One can show without difficulty that f induces homotopy equivalences $X \to A$, $Y \to B$, $X \cap Y \to A \cap B$. It follows from the *spectral sequence* of [1] that f is a homotopy equivalence. ∎

Proof of Theorem 3. Define a map $F_M * F_N \xrightarrow[h]{} F_{M \times N}$ as follows: first, let $PM = \{\alpha \in M^I \mid \alpha(0) = *\}$. Let $\alpha_s \in PM$ be defined by $\alpha_s(t) = \alpha(t - ts)$. Note that $\alpha_1 = *$ and $\alpha_0 = \alpha$. Consider $F_M * F_N$ as a quotient of the space of triples (α, t, β) with $\alpha \in F_M$, $\beta \in F_N$, $-1 \leq t \leq 1$. Define

$$h(\alpha, t, \beta) = \begin{cases} (\alpha_{|t|}, \beta) & \text{if } -1 \leq t \leq 0, \\ (\alpha, \beta_t) & \text{if } 0 \leq t \leq 1 . \end{cases}$$

Using the spectral sequence of [1] in the obvious way, one sees that h is a homotopy equivalence.

Proof of Theorem 1. Let $G: M \to N$ be a homotopy equivalence. If r is large, we may deform g to a proper imbedding in $I^r \times N$. It follows from the relative h-cobordism theorem that $I^r \times N$ is the total space of a k-disk bundle over M. We now appeal to Theorems 6 and 3 to obtain a homotopy equivalence

$$S^{k-1} * F_M \to S^{r-1} * F_N . \quad ∎$$

REFERENCES

1. M. Artin and B. Mazur, *On the van Kampen theorem*. Topology 5 (1966), 179-189.
2. M. Spivak, *Spaces satisfying Poincaré duality*. Topology 6 (1967), 77-101.

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