A NOTE ON MANIFOLDS WHOSE HOLONOMY GROUP IS A SUBGROUP OF $Sp(n) \cdot Sp(1)$

Alfred Gray

Recently, several authors [2], [3], [5] have studied quaternionic analogues of Kähler manifolds. There are two possible definitions of such a manifold. Let M be a connected Riemannian manifold of dimension 4n. Then one can require that the holonomy group H(M) be a subgroup of either Sp(n) or $Sp(n) \cdot Sp(1)$, where $Sp(n) \cdot Sp(1) = Sp(n) \times Sp(1)/(\pm \text{ identity})$.

The condition $H(M) \subseteq Sp(n)$ is equivalent to the existence on M of parallel, globally defined, almost complex structures I, J, and K that satisfy $IJ = -JI = K$. At first glance, it appears that the condition $H(M) \subseteq Sp(n)$ is the more natural generalization of the notion of Kähler manifold. However, it turns out that if $H(M) \subseteq Sp(n)$, then M has Ricci curvature zero. For this reason there are no known examples of compact Riemannian manifolds that are not flat and satisfy the condition $H(M) \subseteq Sp(n)$.

In this paper we consider Riemannian manifolds with $H(M) \subseteq Sp(n) \cdot Sp(1)$, and we call them *quaternionic Kähler manifolds.* Examples are the quaternionic projective spaces and several other symmetric spaces [5]. We show that this definition of quaternionic Kähler manifolds is equivalent to another, which states that a certain tensor field Q is parallel.

The notion of *quaternionic Kähler submanifold* is defined analogously to that of Kähler submanifold. However, the theory of the former is much simpler than that of the latter, because every quaternionic Kähler submanifold is totally geodesic (Theorem 5). This shows, for example, that the quaternionic analogue of the theory of algebraic varieties is trivial.

First we need the following fact about $Sp(n) \cdot Sp(1)$.

PROPOSITION 1. $Sp(n) \cdot Sp(1)$ *is a maximal Lie subgroup of* $SO(4n)$, *for* $n > 1$.

Proof. Let G_0 be a compact connected Lie subgroup of SO(4n) that contains $Sp(n) \cdot Sp(1)$. Then G_0 is transitive on the unit sphere S^{4n-1}, because $Sp(n) \cdot Sp(1)$ is transitive on S^{4n-1}. If

$$Sp(n) \cdot Sp(1) \subset G_0 \subset SO(4n) \qquad \text{(strict inclusion)},$$

it follows from the classification of connected Lie groups acting transitively and effectively on spheres, that the only possibilities for G_0 are U(n), Spin(7), and Spin(9). We rule out U(n), because Sp(n) is a maximal subgroup of U(n). It is easy to verify that the inclusion $Sp(2) \cdot Sp(1) \subseteq Spin(7)$ is impossible, because both groups have rank 3. Finally, Spin(9) is eliminated because $\dim(Spin(9)) = 36 < 39 = \dim(Sp(4) \cdot Sp(1))$. We conclude that $G_0 = Sp(n) \cdot Sp(1)$.

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To complete the proof, it suffices to show that $Sp(n) \cdot Sp(1)$ is its own normalizer in $SO(4n)$. Let $A \in SO(4n)$ normalize $Sp(n) \cdot Sp(1)$. Since $Sp(n) \cdot Sp(1)$ has no outer automorphisms for $n > 1$, there exists $B \in Sp(n) \cdot Sp(1)$ such that AB^{-1} is in the centralizer of $Sp(n) \cdot Sp(1)$. The ordinary representation of $Sp(n) \cdot Sp(1)$ is absolutely irreducible; hence, by Schur's lemma, $\pm AB^{-1}$ is the identity. Thus $A \in Sp(n) \cdot Sp(1)$. This completes the proof.

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Next we obtain a useful algebraic description of $Sp(n) \cdot Sp(1)$. Let V be a 4n-dimensional real vector space. We may regard V as an n-dimensional right quaternion vector space. It is well known that

$$Sp(n) = \{A \in O(4n) \mid A(xi) = (Ax)i \text{ and } A(xj) = (Ax)j \text{ for all } x \in V\}.$$

Then $Sp(n) \cdot Sp(1)$ is the subgroup of $O(4n)$ generated by $Sp(n)$ and the $\mathbb{R}$-linear maps $x \to xi$ and $x \to xj$.

Define $Qx = xi \wedge xj \wedge xk$ for $x \in V$. As it stands, Q is not a tensor; however, it can be linearized so that it becomes a tensor of type (3, 3).

PROPOSITION 2. $Sp(n) \cdot Sp(1) = \{A \in O(4n) \mid QAx = AQx$ *for all* $x \in V\}$.

Proof. The inclusion of the left-hand side of the equation in the right-hand side can be verified directly. Moreover, $QA = AQ$ implies $\det A \geq 0$. Now Proposition 2 follows from Proposition 1.

PROPOSITION 3. *Let* $\{i', j', k'\}$ *be an orthonormal triple of quaternions that has the same orientation as* $\{i, j, k\}$, *and define* $Q'x = xi' \wedge xj' \wedge xk'$ *for* $x \in V$. *Then* $Q = Q'$.

The proposition follows from a direct calculation. We are now ready to discuss quaternionic manifolds.

Definitions. Let M be a differentiable manifold such that each $p \in M$ has a neighborhood U on which there exist three almost complex structures I, J, K such that $IJ = -JI = K$. Suppose Q is a globally defined tensor field of type (3, 3) such that on each U we have the relation $Qx = Ix \wedge Jx \wedge Kx$ whenever $m \in U$ and $x \in M_m$. Then Q is called a *quaternionic structure* on M. If, in addition, M is a Riemannian manifold, if $\|Qx\| = \|x\|^3$, and if Q is parallel, we say that Q is a *quaternionic Kähler structure* on M.

Thus the notions of quaternionic structure and quaternionic Kähler structure are analogous to the notions of almost complex structure and Kähler structure, respectively. From Proposition 3 it follows that a quaternionic structure is well-defined. Furthermore, Proposition 2 implies the following proposition.

PROPOSITION 4. *Let* M *be a* 4n-*dimensional Riemannian manifold. Then*

(i) M *has a compatible quaternionic structure* Q (*that is,* $\|Qx\| = \|x\|^3$) *if and only if the structure group* $O(4n)$ *of the frame bundle of* M *can be reduced to* $Sp(n) \cdot Sp(1)$;

(ii) M *has a compatible quaternionic Kähler structure* Q *if and only if the holonomy group* $H(M)$ *is a subgroup of* $Sp(n) \cdot Sp(1)$.

We note that the existence of a quaternionic Kähler structure does not imply that $H(M) \subseteq Sp(n)$. This is a consequence of the following formula for the covariant derivative of Q:

$$\nabla_X(Q)Y = \nabla_X(I)Y \wedge JY \wedge KY + IY \wedge \nabla_X(J)Y \wedge KY + IY \wedge JY \wedge \nabla_X(K)Y$$

for vector fields X and Y on M. Furthermore, in [2] we showed that quaternionic projective space possesses a quaternionic Kähler structure. Our notion of quaternionic Kähler structure is equivalent to that of Kraines [3] and Wolf [5].

Next we define the quaternionic analogue of an almost complex submanifold.

Definition. Let M and $\overline{M}$ be differentiable manifolds, and let Q be a quaternionic structure on $\overline{M}$. We say that M is a *quaternionic submanifold* of $\overline{M}$ (with respect to Q) provided $Qx \in \Lambda^3 M_m$, for each $m \in M$ and $x \in M_m$. We again denote the induced quaternionic structure on M by Q.

The next theorem shows that the theory of quaternionic Kähler submanifolds is much simpler than that of ordinary Kähler submanifolds.

THEOREM 5. *Let* $\overline{M}$ *be a quaternionic Kähler manifold, and suppose* M *is a quaternionic submanifold of* $\overline{M}$. *Then, with the induced Riemannian structure on* M, M *is a quaternionic Kähler manifold and is totally geodesic in* $\overline{M}$.

Proof. Let ∇ and $\overline{\nabla}$ denote the Riemannian connections of M and $\overline{M}$, and denote by T the configuration tensor of M in $\overline{M}$ (see [1]). Then, for vector fields X and Y, we have the relation

$$\overline{\nabla}_X(Q)Y = \nabla_X(Q)Y + T_X(I)Y \wedge JY \wedge KY + IY \wedge T_X(J)Y \wedge KY + IY \wedge JY \wedge T_X(K)Y.$$

By assumption, the left-hand side of this equation vanishes. Furthermore, each of the four terms on the right-hand side is orthogonal to the others. Hence M is a quaternionic Kähler manifold, and in addition,

$$T_X(I)(Y) = T_X(J)(Y) = T_X(K)(Y) = 0.$$

This implies that

$$T_X Y + T_{IX} IY = T_X Y + T_{JX} JY = T_{IX} IY + T_{JX} JY = 0;$$

therefore $T_X Y = 0$. Thus M is totally geodesic in $\overline{M}$. The theorem generalizes a result of [1].

We conclude by giving some necessary conditions in terms of characteristic classes for the existence of a quaternionic structure.

THEOREM 6. *Let* M *be a* CW-*complex, and let* $\xi = (E, M, p, R^{4n})$ *be a vector bundle. Then*

(i) *a necessary condition for the structure group* O(4n) *of* ξ *to be reduced to* $Sp(n) \cdot Sp(1)$ *is that the Stiefel-Whitney classes* $w_i(\xi)$ *vanish whenever* i *is different from* 2 *and* 3 *and is not divisible by* 4;

(ii) *a necessary condition for the structure group* O(4n) *of* ξ *to be reduced to* Sp(n) *is that the Stiefel-Whitney classes* $w_i(\xi)$ *vanish for all* i *not divisible by* 4.

We omit the proof, which is similar to the corresponding theorem for almost complex structures. (See [4, p. 212].)

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University of California
Berkeley, California 94720
and
University of Maryland
College Park, Maryland 20742