THE ANALYSIS OF REPRESENTATIONS INDUCED FROM A NORMAL SUBGROUP

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1. INTRODUCTION

Following the lines laid down by Clifford in [2], a number of authors have investigated the relation between representations of a group and representations of a normal subgroup. (See the standard reference [5] and the references listed there.) Recently, S. B. Conlon [3] and P. A. Tucker ([9] to [12]) have studied the decomposition of representations induced from indecomposable (and irreducible) representations of a normal subgroup, primarily through an analysis of the endomorphism ring of the induced representation in terms of the endomorphism ring of the original representation.

In the present paper we seek to enlarge and simplify these results. The relation of a group to a normal subgroup is extended somewhat to a situation involving an algebra and a subalgebra (Section 2). Associated with this algebra is a group analogous to the quotient of a group by its normal subgroup. In Section 4 we investigate a particular case in which the commuting ring of the induced module contains a crossed-product of this associated group with a division algebra in the commuting ring of the original module. The induced module turns out to be a free module over the crossed-product, and in the last three sections of the paper we use this to obtain results of Conlon and Tucker as well as some new theorems. These results give a fairly complete picture in the case where the original module is irreducible (Section 7). No restriction is made on the underlying field.

The principal results of the paper are Theorem 2, the propositions of Section 5, and Theorems 4 and 5.

In a recent paper [4], Conlon gives a more functorial approach to the relationship between submodules of an induced module and left ideals in the endomorphism ring of the induced module. (See especially Section 2.3 of [4]. There are some restrictions on the base field, and the objects of study are group rings. In some unpublished work, E. C. Dade has studied algebras axiomatized as in Section 2 of the present paper.)

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2. DEFINITIONS AND PRELIMINARY RESULTS

Let K be a field, and let A be a finite-dimensional algebra over K, with an identity. In [14], K. Yamazaki introduced the concept of *ring extension*. Generalizing this idea, we make the following assumptions about A: we assume that there exists a collection of nonzero subspaces A_g of A (where the index g ranges over a finite group G with identity 1) such that

(1) $A_g A_h = A_{gh}$ $\quad (g, h \in G)$,

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(2) $A_g = a_g A_1 = A_1 a_g$ for some $a_g \in A$,

(3) A is the direct sum of the A_g,

(4) The identity 1 of A is contained in A_1 (so that A_1 is a subalgebra).

The prototype for such an algebra is the following: Let G_0 be a finite group, and let E be a field on which G_0 acts as a group of automorphisms (but not necessarily faithfully). Let K be the subfield of E that is fixed pointwise by all members of G_0, and let A be any crossed-product of G_0 and E corresponding to the action of G_0 on E (see [13]). The representations of A correspond to certain semilinear projective representations of G_0 (see [8]). Now let N_0 be any normal subgroup of G_0, and let $G = G_0/N_0$. If $g \in G$ corresponds to the coset $g_0 N_0$, let A_g be the E-subspace of A spanned by $g_0 N_0$. Then, if A is considered as an algebra over K, the above assumptions are met by these subspaces. When K = E, we have the case of a twisted group algebra and projective representations of G_0.

In the general case, then, A_1 is to be thought of as the analogue of a normal subgroup. For reference, we single out a number of consequences of the assumptions. Concerning relevant facts on tensor products, see [1].

The element a_g is a unit of A and therefore a free generator of A_g as either a left or a right A_1-module. The algebra A is then itself free as either a left or a right A_1-module, and the elements a_g $(g \in G)$ constitute a free basis.

Let M be a left A_1-module (all modules are to be unitary and of finite K-dimension). The *induced module* M^A is defined as $A \otimes_{A_1} M$, where the action of A is on the first factor. Because A is the direct sum of the A_g, the A_1-modules $A_g \otimes_{A_1} M$ are canonically injected into M^A (as A_1-modules), and M^A is their direct sum. Let $M_g = A_g \otimes_{A_1} M$, and identify M with M_1.

The map $m \to a_g \otimes m$ is a K-isomorphism of M with M_g, so that if $|X : K|$ stands for the K-dimension of the K-space X, $|M^A : K| = |G|\,|M : K|$ ($|G|$ is the order of G). In addition, the map produces a lattice isomorphism of A_1-submodules. In particular, M_g is irreducible or indecomposable, according as M is irreducible or indecomposable.

If L and M are A_1-modules and $L \subseteq M$, then the canonical map of L^A into M^A is an injection, because A is a free A_1-module. In fact, if M is a fixed A_1-module, the correspondence $L \to L^A$ of A_1-submodules of M to A-submodules of M^A is a lattice injection.

Let H be a subgroup of G, and let $B = \sum_{g \in H} A_g$. Then B is a subalgebra of A, and it satisfies the same assumptions as A, with H in place of G. If S is a set of left coset representatives of H in G, the elements a_g $(g \in S)$ form a free basis of A as a right B-module. If M is an A_1-module, the B-submodule $BM = \sum_{g \in H} M_g$ of M^A is isomorphic to M^B; and if L is a B-submodule of BM, then AL is isomorphic to L^A $(= A \otimes_B L)$.

3. THE STRUCTURE OF THE COMMUTING RING OF AN INDUCED MODULE

Let A be an algebra satisfying the assumptions in Section 2, and let M be an A_1-module. Then the K-algebra $C = \mathrm{Hom}_A(M^A, M^A)$ of A-endomorphisms of M^A is analyzed by the following lemma (see [3] for the development of the present section):

LEMMA 1 (reciprocity law). *Let* M *be an* A_1*-module and* L *an* A*-module. Then there exists a* K*-isomorphism of* $\mathrm{Hom}_{A_1}(M, L)$ *with* $\mathrm{Hom}_A(M^A, L)$*, given by* $\phi \to \phi^*$*, where for* $\phi \in \mathrm{Hom}_{A_1}(M, L)$*,* ϕ^* *is defined by*

$$(a \otimes m)\phi^* = a(m\phi) \qquad (a \in A,\ m \in M).$$

Proof. It is obvious that $\phi^* \in \mathrm{Hom}_A(M^A, L)$ and that $\phi \to \phi^*$ is K-linear. If $\phi^* = 0$, then (with M and M_1 identified) $m\phi = m\phi^* = 0$ for all $m \in M$; and if $\psi \in \mathrm{Hom}_A(M^A, L)$, then $\psi = \phi^*$, where $\phi = \psi \mid M$ ($\mid$ for restriction). Thus the map is a bijection, as was to be proved. We note in particular that $\phi^* \mid M = \phi$.

THEOREM 1. *Let* M *be an* A_1*-module, and let* $C = \mathrm{Hom}_A(M^A, M^A)$*. For* $g \in G$*, let* $C_g = \{\phi \in C \mid M\phi \subseteq M_g\}$*. Then* C *is the direct sum of the subspaces* C_g*. Moreover:*

(1) $M_g C_h \subseteq M_{gh}$ $\quad (g, h \in G)$.

(2) $C_g C_h \subseteq C_{gh}$ $\quad (g, h \in G)$.

(3) *Let* $i_g\colon M_g \to M^A$ *be the inclusion (an* A_1*-homomorphism). For* $\phi \in \mathrm{Hom}_{A_1}(M, M_g)$*, let* $\phi^A = (\phi i_g)^*$ (see Lemma 1). *Then the map* $\phi \to \phi^A$ *is a* K-*isomorphism of* $\mathrm{Hom}_{A_1}(M, M_g)$ *and* C_g*, and when* $g = 1$*, it is an isomorphism of algebras.*

Proof. (1) and (2) follow from the fact that $M_{gh} = A_g M_h$. Since M^A is A-generated by M, a member of C is determined by its action on M. But because M^A is the A_1-direct sum of the M_g, this implies that $\sum_{g \in G} C_g$ is direct. Let $\pi_g\colon M^A \to M_g$ be the A_1-projection associated with the decomposition of M^A as the direct sum of the M_g. Then, if $\phi \in C$, we define ϕ_g to be $((\phi \mid M)\pi_g i_g)^*$, which is in C_g. Since $m\phi = \sum_{g \in G} m\phi_g$ for $m \in M$, it follows that

$$\phi = \sum_{g \in G} \phi_g \quad \text{and} \quad C = \sum_{g \in G} C_g .$$

The map in (3) has the map $\psi \to (\psi \mid M)\pi_g$ as an inverse, and this establishes the assertions there.

It is easily verified that ϕ is an isomorphism if and only if ϕ^A is an isomorphism.

Consider again a subgroup H of G, and let $B = \sum_{g \in H} A_g$. If M is an A_1-module, identify M^B and $\sum_{g \in H} M_g$, as before. Then Theorem 1, especially part (3), applied to B, will show that with this identification, restriction of $\sum_{g \in H} C_g$ to M^B produces an isomorphism of $\sum_{g \in H} C_g$ and $\mathrm{Hom}_B(M^B, M^B)$.

4. A SPECIAL CASE

The results of the investigations of [12], properly extended, can serve as the tool for a more thorough analysis of induced modules, particularly when the module from which they are induced is irreducible. We shall maintain the notation of Sections 2 and 3.

Let M be an A_1-module, and define G_M as the set

$$G_M = \{g \in G \mid M \text{ and } M_g \text{ are } A_1\text{-isomorphic}\}.$$

G_M is a subgroup of G, because $g \in G_M$ if and only if C_g contains a unit (by the remark after Theorem 1). G_M is called the *inertial group* of M.

Let H be a subgroup of G_M. We shall say that M and H satisfy condition A if the following holds: There exist a division subalgebra D_1 in C_1 having the same identity as C_1, and units $\phi_g \in C_g$ for all $g \in H$, such that the subalgebra D of C generated by D_1 and the ϕ_g is exactly $\sum_{g \in H} D_1 \phi_g$.

D_1 corresponds to a subalgebra of the A_1-endomorphism ring of M, and ϕ_g corresponds to an A_1-isomorphism of M and M_g. The condition amounts to the requirement that for each g and h in H

$$\phi_g \phi_h (\phi_{gh})^{-1} \in D_1 \qquad \text{and} \qquad \phi_g D_1 \phi_g^{-1} = D_1$$

($D_g = D_1 \phi_g \subseteq C_g$). D is a crossed-product of H and D_1. The main result is this:

THEOREM 2. *Let* M *and* H *satisfy condition* A. *Then* M^A *is a free* D-*module. In fact, if* $m_1, \cdots, m_n$ *constitute a (right)* D_1-*basis of* M *(identified with* M_1*) and* S *is a set of left coset representatives of* H *in* G, *then the elements* $a_g m_i$ ($g \in S$, $1 \leq i \leq n$) *form a* D-*basis of* M^A.

Proof. M^A is the direct sum of the subspaces $a_g M_h$ ($g \in S$, $h \in H$), and $M_h = M\phi_h \subseteq MD$. Since $M = \sum_i m_i D_1$, $MD \subseteq \sum_i m_i D$. Therefore $M^A = \sum a_g m_i D$, where the sum is taken over the $g \in S$ and the i with $1 \leq i \leq n$.

Suppose $\sum (a_g m_i)\psi_{ig} = 0$ is a dependence, where $\psi_{ig} \in D$. Therefore $\sum_g a_g \left(\sum_i m_i \psi_{ig}\right) = 0$, and since each inner sum is in $\sum_{h \in H} M_h$, each one is 0. But if $\sum_i m_i \psi_i = 0$ ($\psi_i \in D$), then each ψ_i is 0. To see this, write

$$\psi_i = \sum_h \phi_{ih} \phi_h \qquad (\phi_{ih} \in D_1),$$

where the sum is taken over $h \in H$. Then $\sum_h \left(\sum_i m_i \phi_{ih}\right) \phi_h = 0$. Each inner sum is in M, and $M\phi_h = M_h$. Therefore each inner sum must be 0, and the independence of the m_i then implies that each ϕ_{ih} is 0. In each case, we use the fact that $\sum_{g \in G} M_g$ is direct.

THEOREM 3. *Let* M *and* H *satisfy condition* A. *Then the map* $I \to M^A I$ *of left ideals of* D *to* A-*submodules of* M^A *is a lattice injection. Moreover,*

$$|M^A I : K| = |M : D_1|\,|G : H|\,|I : K| = |M : K|\,|G : H|\,|I : D_1|.$$

Proof. The module $M^A \otimes_D D$, made an A-module by action on the first factor, is isomorphic to M^A by the map $m \otimes \phi \to m\phi$ ($m \in M^A$, $\phi \in D$). If I is a left ideal of D, the canonical map $M^A \otimes_D I \to M^A \otimes_D D$ is an injection, since M^A is a free D-module. In fact, the resulting map from the set of left ideals of D to submodules of $M^A \otimes_D D$ is a lattice injection (see Section 2). The image of $M^A \otimes_D I$ in M^A is precisely $M^A I$.

Because of Theorem 2 and the fact that $I \subseteq D$, $M^A I$ is the direct sum of the K-spaces $a_g m_i I$ ($g \in S$, $1 \leq i \leq n$). This implies the statement on dimensions.

Again, let $B = \sum_{g \in H} A_g$. Then $\sum_{g \in H} M_g = BM$ and $(BM)D \subseteq BM$. Thus $M^A I = A((BM)I)$, and $M^A I$ is therefore isomorphic to $((BM)I)^A$, a module induced from B. Thus, if $H \neq G$, all these modules $M^A I$ are induced.

5. INDECOMPOSABLE MODULES

Let M be a (nonzero) indecomposable A_1-module (with A_1, A, and G as in the previous sections), so that each M_g is also indecomposable. Let G_M be the inertial group of M, and set

$$A_M = \sum_{g \in G_M} A_g, \qquad C_M = \sum_{g \in G_M} C_g.$$

Let R_g be the set of nonunits in C_g. Because C is an endomorphism ring of a finite-dimensional space, the existence of a one-sided inverse is enough to make an element a unit. Moreover, for $g \notin G_M$, it is clear that $C_g = R_g$. Since C_1 is isomorphic to the completely primary ring of A_1-endomorphisms of M, R_1 is the radical of C_1 (see [5, Section 54]).

PROPOSITION 1. *With the above notation,* $R = \sum_{g \in G} R_g$ *is a nilpotent ideal of* C.

Proof. If $g \in G_M$, C_g contains a unit ϕ_g, and $C_g = \phi_g C_1 = C_1 \phi_g$. Hence, $R_g = \phi_g R_1 = R_1 \phi_g$. Since R_1 is a K-subspace of C_1, $\phi_g R_1$ is also a K-subspace. Therefore R is a K-subspace of C. Since $C_g R_h \subseteq R_{gh}$ and $R_h C_g \subseteq R_{hg}$, R is an ideal of C.

Suppose $\phi \in R_g$ and g has order q. Then $\phi^q \in R_1$, and ϕ^q is nilpotent; hence ϕ is nilpotent. Therefore R (as an algebra) is spanned by nilpotent elements, and by a theorem of Wedderburn [5, p. 206] it is itself nilpotent.

Now let $R_M = \sum_{g \in G_M} R_g = C_M \cap R$. Then the kernel of the restriction to C_M of the natural map of C onto C/R is exactly R_M, and C/R is isomorphic to C_M/R_M. It follows (see [6, p. 71]) that if $1 = \varepsilon_1 + \cdots + \varepsilon_r$ is a decomposition of the identity in C_M into orthogonal primitive idempotents, it is also such a decomposition in C. The remarks at the ends of Sections 2 and 3, together with the fact that $A((A_M M)\varepsilon_i) = (AM)\varepsilon_i$, imply the following proposition.

PROPOSITION 2. *Let* M *be an indecomposable* A_1*-module, and let* G_M *be the stability group of* M. *Set* $A_M = \sum_{g \in G_M} A_g$. If M^{A_M} *is decomposed into the direct sum* $L_1 + \cdots + L_r$ *of indecomposable* A_M*-modules, then* M^A *is isomorphic to the direct sum* $L_1^A + \cdots + L_r^A$, *and the* L_i^A *are indecomposable.*

As a preliminary for the next result, let $C' = C/R$ and let ' denote the images under the natural map of C to C'. We see that $C_g \cap R = R_g$ and that $C'_g = 0$ if $g \notin G_M$. Therefore C' is the direct sum of the spaces C'_g ($g \in G_M$), and $C'_g C'_h \subseteq C'_{gh}$. Furthermore, C'_1 is isomorphic to the division algebra C_1/R_1. If $\psi_g \in C_g$ is a unit, where $g \in G_M$, then ψ'_g is a unit of C'. Thus $C' = \sum_{g \in G_M} C'_1 \psi'_g$.

LEMMA 2. *Suppose that* $R \neq 0$ *and that* t *is the smallest integer for which* $R^t = 0$. *For* $1 \leq i \leq t+1$, *let*

$$M_i = \{m \in M \mid mR^{t+1-i} = 0\} \qquad (\textit{with } R^0 = C).$$

Then each M_i *is an* A_1-*module, and*

(1) $AM_i = M_i^A = \{m \in M^A \mid mR^{t+1-i} = 0\}$;

(2) $M = M_1 \supset M_2 \supset \cdots \supset M_{t+1} = 0$, *each inclusion being proper;*

(3) $M_i^A C \subseteq M_i^A$ *and* $M_i^A R \subseteq M_{i+1}^A$;

(4) C' *acts faithfully on* M_i^A/M_{i+1}^A *by means of the action*

$$(m + M_{i+1}^A)\phi' = m\phi + M_{i+1}^A.$$

Proof. First of all, $(AM_i)R^{t+1-i} = 0$. Suppose that

$$\Big(\sum_{g \in G} a_g m_g\Big) R^{t+1-i} = 0, \qquad \text{where } m_g \in M.$$

R^{t+1-i} is additively generated by the $(t+1-i)$-fold products of the members of the R_h $(h \in G)$. If $r = r_1 \cdots r_{t+1-i}$ is such a product, where $r_j \in R_{h_j}$, then $r \in C_h$ $(h = h_1 \cdots h_{t+1-i})$. The relation $\Big(\sum_{g \in G} a_g m_g\Big) r = 0$ implies that $m_g r = 0$ for each g. Therefore $m_g R^{t+1-i} = 0$ and $m_g \in M_i$. Thus $\sum_{g \in G} a_g m_g \in AM_i$, and (1) holds.

Now $M_{i+1} \subseteq M_i$, and if $M_{i+1} = M_i$, then $M_i^A = M_{i+1}^A$. But $M^A R^{i-1} \subseteq M_i^A$, and therefore

$$M^A R^{i-1} \subseteq M_{i+1}^A \qquad \text{and} \qquad M^A R^{t-1} = 0,$$

which is impossible. Thus (2) holds. (3) is evident, and (3) implies that the action defined for C' in (4) is legitimate. Suppose that $\phi' \neq 0$ but $M_i^A \phi \subseteq M_{i+1}^A$. Replacing ϕ by $\psi_g \phi$ for an appropriate g, we may assume that the C_1-component ϕ_1 of ϕ is not in R_1. The assumption on ϕ then implies that $M_i \phi_1 \subseteq M_{i+1}$. But since ϕ_1 is invertible, this would mean that $|M_i : K| \leq |M_{i+1} : K|$, contrary to (2). Thus (4) holds.

Continuing with the notation above, form the A_1-module $N_i = M_i/M_{i+1}$, and let $'$ refer to the natural map of M_i onto N_i (in addition to its other uses). Then $'$ extends to an A-homomorphism of M_i^A onto N_i^A with kernel M_{i+1}^A by means of $(a \otimes m)' = a \otimes m'$. If N_i^A is thus identified with M_i^A/M_{i+1}^A, the action of C' on N_i^A is given by $(a \otimes m')\phi' = ((a \otimes m)\phi)'$. Thus

$$N_i C_g' = M_i' C_g' = (M_i C_g)' \subseteq (A_g M_i)' = A_g N_i.$$

Therefore N_i and G_M satisfy condition A of Section 4, with C_1' for D_1, ψ_g' for ϕ_g, and C' for D.

PROPOSITION 3. *Let* ε *be an idempotent of* C_M. *Then*

$$|M^A \varepsilon : K| = |M : K|\,|G : G_M|\,|C'\varepsilon' : C_1'|.$$

Proof. Of course, $M^A \varepsilon = M^A C_M \varepsilon$. If $R = 0$, the result follows from Theorem 3; therefore we may assume $R \neq 0$. The kernel of the composition

$$M_i^A \to N_i^A \to N_i^A \varepsilon'$$

is $M_{i+1}^A \varepsilon + M_i^A(1 - \varepsilon)$, so that the kernel of $M_i^A \varepsilon \to M_i^A \to N_i^A \to N_i^A \varepsilon'$ is

$$M_i^A \varepsilon \cap (M_{i+1}^A \varepsilon + M_i^A(1 - \varepsilon)) = M_{i+1}^A \varepsilon .$$

Therefore $N_i^A \varepsilon'$ is isomorphic to $M_i^A \varepsilon / M_{i+1}^A \varepsilon$. But by Theorem 3,

$$|N_i^A \varepsilon' : K| = |N_i^A C' \varepsilon' : K| = |N_i : K| \, |G : G_M| \, |C' \varepsilon' : C_1'| .$$

Addition of these dimensions for $1 \leq i \leq t$ gives the result.

For the sake of completeness, we add the following fact (see [7]):

PROPOSITION 4. *Let* ε_1 *and* ε_2 *be two idempotents in* C_M. *Then the modules* $M^A \varepsilon_1$ *and* $M^A \varepsilon_2$ *are isomorphic if and only if* $C_M \varepsilon_1$ *and* $C_M \varepsilon_2$ *are isomorphic* C_M*-modules. That in turn is true if and only if* $C' \varepsilon_1'$ *and* $C' \varepsilon_2'$ *are isomorphic* C'*-modules.*

The aggregate of Propositions 2, 3, and 4 compares with the theorem of Section 2 of [3]. We point out again that C' is a crossed-product of the division algebra C_1' and G_M.

6. INDECOMPOSABLE MODULES UNDER CONDITION A

Continuing the notation of Section 5, let $\bar{M}$ be an indecomposable A_1-module with the following property: M and $H = G_M$ satisfy condition A of Section 4, and C_1 is the K-direct sum of R_1 and the division algebra D_1 of condition A. Situations in which this occurs are discussed in [12]. One particularly important case is that in which M is irreducible, the validity then being a consequence of Schur's lemma. Under the present assumptions, D_1 and D may be identified with the C_1' and C' of Section 5, respectively. Moreover, C is the K-direct sum of D and R.

THEOREM 4. *Under the present conditions, let* I_1 *and* I_2 *be any two left ideals of the algebra* D. *Then there is a* K*-injection of* $\mathrm{Hom}_D(I_1, I_2)$ *into* $\mathrm{Hom}_A(M^A I_1, M^A I_2)$ *as a* K*-direct summand, such that if* i *and* π *stand uniformly for the injection and an appropriate projection, both maps are functorial: if* I_1, I_2, *and* I_3 *are three left ideals of* D *and*

$$x_1 \in \mathrm{Hom}_D(I_1, I_2) \quad \textit{and} \quad x_2 \in \mathrm{Hom}_D(I_2, I_3),$$

then $i(x_1 x_2) = i(x_1) i(x_2)$. *Similarly, if*

$$y_1 \in \mathrm{Hom}_A(M^A I_1, M^A I_2) \quad \textit{and} \quad y_2 \in \mathrm{Hom}_A(M^A I_2, M^A I_3),$$

then $\pi(y_1 y_2) = \pi(y_1)\pi(y_2)$.

Proof. Section 4 implies that if I is a left ideal of D, then $M^A \otimes_D I$, made an A-module by action on the first factor, is isomorphic to $M^A I$. Therefore, for each $x \in \mathrm{Hom}_D(I_1, I_2)$, we define $i(x)$ to be the map $1 \otimes x$, where 1 is the identity map of M^A. The functorial property for i is then obvious. In addition, because M^A is a free D-module, $x \neq 0$ implies that $1 \otimes x \neq 0$.

For any left ideal I of D, let

$$I^* = \{\phi \in C \mid M^A\phi \subseteq M^A I\} = \{\phi \in C \mid M\phi \subseteq M^A I\}.$$

I^* is a left ideal of C. By Lemma 2, there exists an $m_0 \neq 0$ in M such that $m_0 R = 0$, and by Theorem 2, m_0 can be incorporated in a D-basis of M^A. If $\phi \in I^*$ and $\phi = \phi_1 + \phi_2$, with $\phi_1 \in D$ and $\phi_2 \in R$, then $m_0\phi = m_0\phi_1 \in M^A I$. But on using a D-basis containing m_0 and writing out this member of $M^A I$, we conclude that $\phi_1 \in I$. Therefore $\phi_2 \in I^*$. Thus $I^* = I + (I^* \cap R)$ (and this is actually a D-direct sum).

Let now $y \in \mathrm{Hom}_A(M^A I_1, M^A I_2)$, and consider the composition $m \to m\phi \to (m\phi)y$ $(m \in M)$ for a fixed ϕ in I_1^*. This is an A_1-homomorphism of M into M^A, and by reciprocity (Lemma 1) there exists a member ϕ^y of C such that $(m\phi)y = m\phi^y$. This relation persists for all m in M^A. The map $\phi \to \phi^y$ is K-linear, and if $\psi \in C$, then

$$m(\psi\phi)^y = (m(\psi\phi))y = ((m\psi)\phi)y = (m\psi)\phi^y \quad \text{for all } m \in M,$$

so that $(\psi\phi)^y = \psi\phi^y$. Moreover, ϕ^y is in I_2^*, because $(m\phi)y \in M^A I_2$ for all $m \in M$.

Therefore the map y', given by $\phi \to \phi^y$, is in $\mathrm{Hom}_C(I_1^*, I_2^*)$. Let π_j stand for the D-projection of $I_j^* = I_j + (I_j^* \cap R)$ onto I_j. We define the map

$$\pi\colon \mathrm{Hom}_A(M^A I_1, M^A I_2) \to \mathrm{Hom}_D(I_1, I_2)$$

as follows: for $y \in \mathrm{Hom}_A(M^A I_1, M^A I_2)$, $\pi(y)$ is the restriction of the map $\phi \to (\phi^y)\pi_2$ to I_1. The map π is K-linear, and if $x \in \mathrm{Hom}_D(I_1, I_2)$ is written as an exponent, then

$$m(\phi^{1\otimes x}) = (m\phi)(1 \otimes x) = m\phi^x,$$

for $\phi \in I_1$ and $m \in M$. Thus $\pi i(x) = x$.

It remains to establish the functorial nature of π. Suppose

$$y \in \mathrm{Hom}_A(M^A I_1, M^A I_2) \quad \text{and} \quad z \in \mathrm{Hom}_A(M^A I_2, M^A I_3).$$

Then

$$m\phi^{yz} = (m\phi)yz = ((m\phi)y)z = (m\phi^y)z = m(\phi^y)^z \quad \text{for } m \in M \text{ and } \phi \in I_1^*.$$

Therefore, $(yz)' = y'z'$. With π_j as above, we then have the relation

$$(\phi^{yz})\pi_3 = [((\phi^y)\pi_2)^z]\pi_3 + [((\phi^y)(1 - \pi_2))^z]\pi_3.$$

If we show that $(I_2^* \cap R)^z \subseteq I_3^* \cap R$, then the second term will be 0 and $(\phi^{yz})\pi_3 = [((\phi^y)\pi_2)^z]\pi_3$; that is, $\pi(yz) = \pi(y)\pi(z)$. Again, let $m_0 \in M$, $m_0 \neq 0$, with $m_0 R = 0$. Since $C = D + R$ is direct, R is exactly the annihilator of m_0, because m_0 is a member of a D-basis of M^A. If $\psi \in I_2^* \cap R$, then $m_0\psi = 0$ and $m_0\psi^z = 0$. Therefore $\psi^z \in I_3^* \cap R$.

Thus the proof of Theorem 4 is complete. If $R = 0$ and if $\phi^y = 0$ for all ϕ in I_1, then $(M^A I_1)y = 0$. This gives the following result.

COROLLARY 1. *If* $R = 0$, *then the kernel of* π *is* 0.

In the case $I_1 = I_2 = I$, both i and π are ring homomorphisms, and by the theorem they preserve the identities. Since also $\mathrm{Hom}_A(M^A I, M^A I)$ is the K-direct sum of $i(\mathrm{Hom}_D(I, I))$ and the kernel of π, and since the identity is in the first summand, the kernel of π consists of nonunits. Therefore either both or neither of the two rings is completely primary.

COROLLARY 2. *If* I *is a left ideal of* D, *then* $M^A I$ *is indecomposable if and only if* I *is indecomposable. Furthermore, if* I_1 *and* I_2 *are left ideals in* D, *then* $M^A I_1$ *and* $M^A I_2$ *are isomorphic if and only if* I_1 *and* I_2 *are isomorphic. In fact,* I_1 *is isomorphic to a direct summand of* I_2 *if and only if* $M^A I_1$ *is isomorphic to a direct summand of* $M^A I_2$.

The second and third assertions follow from the functorial nature of π and i. It is clear how Corollary 2 will match a direct decomposition of D with one of M^A. We wish now to prove that for $I_1 = I_2 = I$, the kernel of π is nilpotent.

LEMMA 3. *If* I *is a left ideal of* D, *then for any exponent* e,

$$M^A I \cap M^A R^e = M^A R^e I.$$

Proof. Since $R = \sum_{g \in G} (R \cap C_g)$ (Section 5) the analogous formula holds for R^e: $R^e = \sum_{g \in G} (R^e \cap C_g)$. Thus

$$M^A R^e = \sum_{g \in G} (M^A R^e \cap M_g).$$

Let $V_g = M^A R^e \cap M_g$. Then $a_h V_g = V_{hg}$ (a_h as in Section 2). If $h \in G_M$, then C_h contains a unit ϕ_h of D (by condition A), and $V_g \phi_h = V_{gh}$. V_g is a D_1-space, so that $M_1 = V_1 + W_1$, where W_1 is a complementary D_1-space to V_1. Let S be a set of left coset representatives of G_M in G, and put

$$V_M = \sum_{h \in G_M} V_1 \phi_h, \qquad W_M = \sum_{h \in G_M} W_1 \phi_h.$$

Then it follows from the preceding remarks and Theorem 2 that M^A is the D-direct sum of $\sum_{g \in S} a_g V_M$ and $\sum_{g \in S} a_g W_M$, and that $M^A R^e = \sum_{g \in S} a_g V_M$. Therefore, because $I \subseteq D$, $M^A I \cap M^A R^e$ must be in $\left(\sum_{g \in S} a_g V_M\right) I$, that is, in $M^A R^e I$. Since the reverse inclusion is automatic, the result follows.

PROPOSITION 5. *If* I *is a left ideal of* D, *then the kernel of the homomorphism* π: $\mathrm{Hom}_A(M^A I, M^A I) \to \mathrm{Hom}_D(I, I)$ *is nilpotent.* (*If* $R = 0$, *the kernel is* 0, *by* Corollary 1 *of* Theorem 4.)

Proof. We may assume that $R \neq 0$. The kernel of π is the set of elements y in $\mathrm{Hom}_A(M^A I, M^A I)$ such that, in the notation of the proof of Theorem 4, $\phi^y \in I^* \cap R$ for all $\phi \in I$. For such a y,

$$(m\phi)y = m\phi^y \in M^A R \cap M^A I = M^A RI \qquad (m \in M^A,\ \phi \in I).$$

Suppose that $(M^A I)y^e \subseteq M^A R^e I$. Then

$$(M^A I)y^{e+1} = ((M^A I)y^e)y \subseteq (M^A R^e I)y \subseteq M^A R^e I^y \subseteq M^A R^{e+1}.$$

Since $(M^A I)y^{e+1} \subseteq M^A I$, it follows that

$$(M^A I)y^{e+1} \subseteq M^A R^{e+1} \cap M^A I = M^A R^{e+1} I.$$

Thus, by induction, $(M^A I)y^e \subseteq M^A R^e I$ for all positive integers e.

But now let e be so large that $R^e = 0$ (Proposition 1). Then $y^e = 0$. Hence the kernel of π consists of nilpotent elements and is therefore nilpotent.

The results of this section are to be compared with those of [12] (which in turn relates to the other papers of Tucker).

7. THE IRREDUCIBLE CASE

Keeping the notation of Section 6, let now M be an irreducible A_1-module. Then, as has been pointed out, M and G_M satisfy condition A of Section 4. Since all the M_g are irreducible A_1-modules, it follows from Schur's lemma that $R = 0$. Therefore D and C coincide, and the results of Section 6 carry over directly. In addition, one can identify explicitly the modules $M^A I$, where I is a left ideal of C:

PROPOSITION 6. *Let* M *be an irreducible* A_1*-module, let* G_M *be the inertial group of* M, *and let* $A_M = \sum_{g \in G_M} A_g$ (see Section 5). *Then the submodules of* M^A *of the form* $M^A I$ (I *a left ideal of* C) *are exactly the submodules* AL, *where* L *is an* A_M*-submodule of* $\sum_{g \in G_M} M_g$.

Proof. That $M^A I$ is of this form was remarked at the end of Section 4. Conversely, consider such a submodule AL, and let $I = \{\phi \in C \mid M^A \phi \subseteq AL\}$. Then $\phi \in I$ if and only if $M\phi \subseteq AL$. Since $AL \cap \sum_{g \in G_M} M_g = L$, this means that $\phi \in I$ if and only if $M\phi \subseteq L$. I is a left ideal of C; we shall prove that $MI = L$ (so that $AL = AMI = M^A I$).

Since M is irreducible, $M = A_1 m_0$ for each nonzero m_0 in M. Thus, if $m_0 \phi \in L$ for some $\phi \in C$, then $m\phi \in L$ for all $m \in M$, so that $\phi \in I$. Suppose then that $\sum m_i \phi_i \in L$, where $\phi_i \in C$ and the m_i form a C_1-basis of M (see Theorem 2). By the density theorem for irreducible modules (see [7, p. 28]), one can find for each j an $a_j \in A_1$ with $a_j m_j = m_j$ and $a_j m_i = 0$ $(i \neq j)$. Therefore $m_j \phi_j \in L$ for each j, so that $\phi_j \in I$. Thus $L \subseteq MI$, and since $MI \subseteq L$, the assertion follows.

If in particular $G_M = G$, then every A-submodule of M^A is of the form $M^A I$. Thus in this case Theorem 3, Theorem 4, Corollary 1, and the above imply the following theorem.

THEOREM 5. *Let* M *be an irreducible* A_1*-module such that the inertial group* G_M *of* M *is all of* G. *Then*

(1) *there is a lattice isomorphism between the set of left ideals of* $C = \mathrm{Hom}_A(M^A, M^A)$ *and the* A*-submodules of* M^A *set up by* $I \to M^A I$. *Furthermore,* $|M^A I : K| = |M : K|\, |I : C_1|$;

(2) *the* K*-spaces* $\mathrm{Hom}_C(I_1, I_2)$ *and* $\mathrm{Hom}_A(M^A I_1, M^A I_2)$ *are isomorphic, and the isomorphism is functorial in the sense of Theorem* 4;

(3) *the rings* $\mathrm{Hom}_C(I, I)$ *and* $\mathrm{Hom}_A(M^A I, M^A I)$ *are isomorphic.*

Note that (1) implies that M^A is completely reducible if and only if C is semi-simple. Again we point out that C is a crossed-product of C_1 and G_M ($G_M = G$, here).

We close with a sketch of the Clifford correspondence (see the exposition in Section 51 of [5]). First of all, the reciprocity law in the opposite direction holds: if M is an A_1-module and L is an A-module, then $\mathrm{Hom}_{A_1}(L, M)$ and $\mathrm{Hom}_A(L, M^A)$ are isomorphic. The isomorphism is established by the map $\phi \to \phi'$, where

$$u\phi' = \sum_{g \in G} a_g((a_g^{-1} u)\phi) \quad (u \in L).$$

An analogue of Clifford's theorem [5, p. 343] holds: if L is an irreducible A-module, then the restriction of L to A_1 is completely reducible. If M is an irreducible constituent of this restriction, then the other constituents are of the form $A_g \otimes_{A_1} M$ (conjugates of M).

Now, given an irreducible A-module L, let M be an irreducible A_1-constituent of L. In L, form the A_1-submodule L_0 consisting of the sum of all the A_1-submodules of L isomorphic to M. Then L_0 is an irreducible A_M-module and L is isomorphic to L_0^A. (Here G_M and A_M are defined as in Section 5.) By the reciprocity, L_0 is isomorphic to an A_M-submodule of M^{A_M}. Conversely, if L_0 is any irreducible A_M-submodule of M^{A_M}, then L_0^A is also irreducible.

Thus the determination of the irreducible A-modules consists of a mechanical induction step and an analysis of irreducible submodules of M^{A_M}; but this second step is carried out by means of Theorem 5.

Consider again Theorem 5 and the module $M^A I$. If ϕ_g denotes a unit of C_g, then $M_g = M\phi_g$. Thus $M^A I$ is actually equal to MI. From (1) of Theorem 5, it follows that the K-map $M \otimes_{C_1} I \to MI$ given by $m \otimes \phi \to m\phi$ is a bijection. Thus $M^A I$ may be regarded as the K-space $M \otimes_{C_1} I$ on which the action of A must be defined. But if $a \in A_g$, then, for each $m \in M$, we see that $am \in M_g$ and $(am)\phi_g^{-1} \in M$. Therefore $a(m\phi) = ((am)\phi_g^{-1})\phi_g \phi$. Consequently we obtain the following result.

PROPOSITION 7. *In the case of Theorem* 5, *the* A-*submodules of* M^A *are of form* $M \otimes_{C_1} I$ (I *a left ideal of* C), *where the action of* A *is given by the rule*

$$a(m \otimes \phi) = (am)\phi_g^{-1} \otimes \phi_g \phi \quad \textit{for } a \in A_g.$$

Since C is a crossed-product of G and C_1, this result corresponds to that of Clifford.

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