ON THE GROUP OF AUTOMORPHISMS OF A FINITE-DIMENSIONAL TOPOLOGICAL GROUP

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Let G be a locally compact topological group, and let A(G) be the group of all (bicontinuous) automorphisms of G. There is then a natural topology in A(G) under which A(G) is a topological group. However, this group is not necessarily locally compact. In fact, some otherwise rather well-behaved groups G (such as infinite-dimensional tori) fail to have a locally compact A(G). The main purpose of this work is to show that if G is a connected, locally compact, finite-dimensional topological group, then A(G) is locally compact and is, moreover, a Lie group.

The word *group* will always mean a topological group, and the identity element of a group will be denoted by 1.

1. PRELIMINARIES

Here we collect some standard definitions and some more or less well-known facts.

1.1. The group A(G) corresponding to a locally compact group G is topologized as follows: For a compact subset C of G and a neighborhood U of 1 in G, let W[C, U] be the set of all $\theta \in A(G)$ such that $\theta(x)x^{-1}$ and $\theta^{-1}(x)x^{-1}$ lie in U for all $x \in C$. Then, as C runs through all compact subsets of G, and U through all neighborhoods of 1 in G, the sets W[C, U] form a system of basic neighborhoods of 1 in A(G). Under this topology, A(G) is a topological group.

If G is a compact group, then this topology coincides with the so-called compact open topology, and if moreover G is a Lie group, then this is the same as the relative topology on the subspace A(G) of a general linear group $GL(n, \mathbb{R})$ ($n = \dim G$). In general, however, the topology we defined above is stronger than the compact open topology on A(G). We also remark that if G is connected and locally connected, then the compact subsets C in W[C, U] may be assumed to be connected.

1.2. Let G be a connected, locally compact group. Then G is locally the product of a compact group K and a local Lie group L_ℓ^*, with K and L_ℓ^* normalizing each other. That is, there exists a neighborhood U of 1 in G such that

$$U = K \times L_\ell^* \quad \text{and} \quad [K, L_\ell^*] = \{1\},$$

where, for subsets A and B of G, [A, B] denotes the commutator subgroup of A and B. Since G is connected, the relation $G = KL^*$ holds, where L^* is the subgroup of G defined by

$$L^* = \bigcup_n L_\ell^{*n}.$$

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Clearly, K and L^* are both normal in G; but we also note that L^* is not necessarily closed in G. There exists a connected Lie group structure on L^*, which is uniquely determined by the local Lie group L_ℓ^* and which we shall denote by L. The inclusion map

$$\lambda: L \to G$$

is clearly continuous, and $\lambda(L) = L^*$.

Suppose now that G is, in addition, finite-dimensional. It is well known (see [3], for example), that we can then choose K to be totally disconnected. Let D^* denote the intersection of K and L^*, and put $D = \lambda^{-1}(D^*)$. Then, since $K \cap L_\ell^* = \{1\}$, D is a discrete, closed, normal subgroup of L and hence is central in L. Finally, we note that $\lambda(L) = L^*$ is dense in G.

2. MAIN RESULTS

2.1. THEOREM. *Let* G *be a connected, locally compact, finite-dimensional group. Then there exist a connected Lie group* L *and monomorphisms*

$$\lambda: L \to G \quad \textit{and} \quad \phi: A(G) \to A(L)$$

of topological groups such that, for each $\theta \in A(G)$, *the following diagram commutes:*

$$\begin{array}{ccc} G & \xrightarrow{\theta} & G \\ \lambda \uparrow & & \uparrow \lambda \\ L & \xrightarrow[\phi(\theta)]{} & L \end{array}$$

Proof. Let $U = K \times L_\ell^*$, L^*, and L be related as in Section 1.2, and let $\theta \in A(G)$. Let V be a neighborhood of 1 in G such that $V \cup \theta(V) \subset U$, and consider the local decomposition (as in 1.2)

$$V = K' \times L_\ell'^* .$$

The connectivity of $L_\ell'^*$ implies that $\theta(L_\ell'^*) \subset L_\ell^*$. Since $L_\ell'^*$ is open in L_ℓ^* (and hence in L), the bicontinuous local isomorphism

$$\theta: L_\ell'^* \to L_\ell^*$$

induces an automorphism $\phi(\theta) \in A(L)$, which is uniquely determined by θ. It is clear then that the diagram commutes, for each $\theta \in A(G)$, and that ϕ is a homomorphism (algebraically).

To prove the continuity of ϕ, let W[C, N] be a basic neighborhood of 1 in A(L), where C is a compact, connected subset of L, and where N is a neighborhood of 1 in L. Since L is a Lie group, we may assume that there exists a neighborhood V of 1 in G such that $\lambda(N)$ is the local Lie group that occurs in its decomposition. That is, V can be written as

$$V = K_1 \times \lambda(N),$$

for some totally disconnected, compact group K_1. Since λ is continuous, $\lambda(C)$ is compact and connected. We claim now that the neighborhood $W[\lambda(C), V]$ of the identity in $A(G)$ is mapped into $W[C, N]$ by ϕ. In fact, let $\theta \in W[\lambda(C), V]$. Then, for every $x \in C$, we have the relations

$$\theta\lambda(x)\lambda(x^{-1}) \in V \quad \text{and} \quad \theta^{-1}\lambda(x)\lambda(x^{-1}) \in V .$$

But $x \to \theta\lambda(x)\lambda(x^{-1})$ and $x \to \theta^{-1}\lambda(x)\lambda(x^{-1})$ are continuous functions from the connected set C into V. Hence, $\theta\lambda(x)\lambda(x^{-1})$ and $\theta^{-1}\lambda(x)\lambda(x^{-1})$ are in $\lambda(N)$ for every $x \in C$. Noting that $\lambda\phi(\theta) = \theta\lambda$, we conclude that $\phi(\theta)(x)x^{-1} \in N$ and $\phi(\theta)^{-1}(x)x^{-1} \in N$ for all $x \in C$, and it follows that $\phi(\theta) \in W[C, N]$. Since ϕ is continuous at the identity, ϕ is continuous.

Finally, it remains to show that ϕ is one-to-one. But this follows immediately from the fact that $\lambda(L) = L^*$ is dense in G (see 1.2) and that θ is the identity map on L^* if $\theta \in \operatorname{Ker} \phi$.

2.2. LEMMA. *Let* G *and* $U = K \times L_\ell^*$ *be related as in Section* 1.2, *and let*

$$A'(G) = \{\theta \in A(G): \theta(K) \subset K\} .$$

Then $A'(G)$ *is an open subgroup of* $A(G)$.

Proof. We may assume that L_ℓ^* is so small that it contains no nontrivial compact subgroup. Thus K is the maximal compact subgroup contained in U. Hence $A'(G)$ coincides with the set

$$\mathscr{A} = \{\theta \in A(G): \theta(K) \subset U\} .$$

But the latter is open in $A(G)$, so that $A'(G)$ is open in $A(G)$.

2.3. LEMMA. *Let* $A_D(L)$ (*respectively*, $A_{D^*}(G)$) *be the subgroup of* $A(L)$ (*respectively, of* $A'(G)$) *consisting of all* $\tau \in A(L)$ (*of all* $\theta \in A'(G)$) *that leave every element of* D (*of* D^*) *point-wise fixed. Then* $A_{D^*}(G)$ *is topologically isomorphic with* $A_D(L)$ *under* ϕ. *In particular*, $A_{D^*}(G)$ *is a Lie group.*

Proof. Let $\theta \in A_{D^*}(G)$. Then $\phi(\theta) \in A_D(L)$. To see this, note first that $\phi(\theta) = \lambda^{-1}\theta\lambda$, by Theorem 2.1. Thus

$$\phi(\theta)(d) = \lambda^{-1}\theta\lambda(d) = \lambda^{-1}\lambda(d) = d \qquad \text{for } \theta \in A_{D^*}(G).$$

Hence $\phi(\theta) \in A_D(L)$.

Now let $\tau \in A_D(L)$. Then $\tau(d) = d$, for all $d \in D$. Let θ_τ be defined by

$$\theta_\tau(x) = \begin{cases} x & \text{if } x \in K, \\ \lambda\tau\lambda^{-1}(x) & \text{if } x \in \lambda(L) = L^*, \end{cases}$$

where K and L satisfy the conditions in Section 1.2. Since $\theta_\tau(d^*) = d^*$ for $d^* \in D^*$, $\theta_\tau \in A_{D^*}(G)$. From the definition, it is clear that $\tau \to \theta_\tau$ is continuous and that $\phi(\theta_\tau) = \tau$. Thus $A_{D^*}(G)$ and $A_D(L)$ are isomorphic (topologically) under ϕ.

Since $A_D(L)$ is a closed subgroup of the Lie group $A(L)$, it follows that $A_D(L)$ and $A_{D^*}(G)$ are both Lie groups.

2.4. LEMMA. *Let* $A'(L) = \{\tau \in A(L): \tau(D) \subset D\}$. *Then* $A_D(L)$ *is an open subgroup of* $A'(L)$.

Proof. Since D is a discrete central subgroup of L, D is finitely generated (see [2], for example). Let $C = \{d_1, d_2, \cdots, d_n\}$ be a set of generators of the central subgroup D of L, and let N be a neighborhood of 1 in L such that $D \cap N = \{1\}$. To show that $A_D(L)$ is open in $A'(L)$, it suffices to show that

$$W[C, N] \cap A'(L) \subset A_D(L).$$

Let $\tau \in W[C, N] \cap A'(L)$. Then

$$\tau(d_i)d_i^{-1} \in D \cap N = \{1\} \qquad (1 \le i \le n).$$

Therefore $\tau(d_i) = d_i$ $(1 \le i \le n)$, and since the d_i form a set of generators of D, $\tau(d) = d$ for all $d \in D$. Consequently, $\tau \in A_D(L)$.

2.5. THEOREM. *Let* G *be a connected, locally compact, finite-dimensional group. Then* A(G) *is a Lie group.*

Proof. It suffices to show that the Lie group $A_{D^*}(G)$ is open in A(G). Since A'(G) is open in A(G) (Lemma 2.2), it is then enough to show that $A_{D^*}(G)$ is open in A'(G). To see this, note first that $\phi(A'(G)) \subset A'(L)$. It is also clear that $A_D(L)$ and $A_{D^*}(G)$ are normal in A'(L) and A'(G), respectively. Thus ϕ induces a monomorphism of topological groups:

$$\hat{\phi}: A'(G)/A_{D^*}(G) \to A'(L)/A_D(L).$$

Since $A_D(L)$ is open in A'(L) (Lemma 2.4), it follows that $A'(L)/A_D(L)$ is discrete. Hence the continuity of $\hat{\phi}$ implies that $A'(G)/A_{D^*}(G)$ is discrete, and this proves that $A_{D^*}(G)$ is open in A'(G).

2.6. COROLLARY. *Let* G *be a finite-dimensional, compact, connected abelian group. Then* A(G) *is discrete.*

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