EICHLER INTEGRALS AND THE AREA THEOREM OF BERS

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1. Let Γ be a group of fractional linear transformations acting on the extended complex plane. We denote the limit point set by Λ and the set of discontinuity by Ω. It is assumed that Λ is infinite, and that Ω is not empty.

The orbit space $S = \Omega/\Gamma$ falls into components S_i that inherit the complex structure of the plane. On each component there is an invariant Poincaré metric $ds = \lambda\,|dz|$ with curvature -1.

L. Bers [2] has recently proved the following remarkable fact:

THEOREM (Bers). *If* Γ *can be generated by* N *elements, the total Poincaré area of* S *is at most* $4\pi(N - 1)$.

In this paper we give a different version of Bers' proof. It is based on the same idea, but it uses singular Eichler integrals rather than Beltrami differentials.

Sections 2 to 9 are restatements of known facts in the form that we need. The proof is in Sections 10 to 14, and in Section 15 we show that the number of components is at most $18(N - 1)$, a slight improvement on the bound given by Bers.

2. The projection map $\pi\colon \Omega \to S$ defines a ramification number $n(p) \geq 1$ at every $p \in S$, and the points with $n(p) > 1$ are isolated. They are projections of elliptic fixed points. If the fixed point is placed at 0, for convenience, the projection may be expressed through $\tilde{z} = z^n$, where $n = n(p)$ and $\tilde{z}$ is the value at $\pi(z)$ of a local parameter.

For finitely generated Γ, it had been shown that there are only a finite number of points with $n(p) > 1$ on each S_i. Moreover, S_i can be extended to a compact surface $\overline{S}_i$ by the addition of a finite number of points, and we set $n(p) = \infty$ when $p \in \overline{S}_i - S_i$. In a typical case, the projection near such a point becomes $\tilde{z} = e^{-1/z}$. To a small disk $\tilde{\Delta}$: $(|\tilde{z}| < \delta)$ there corresponds a disk Δ in the z-plane whose center lies on the positive real axis and whose circumference passes through 0. The disk Δ is contained in Ω, and it is mapped upon itself by a parabolic transformation in Γ with fixed point at the origin; all other images of Δ under Γ are disjoint.

The genus of $\overline{S}_i$ is denoted by g_i. We recall that the Poincaré area of S_i is given by

$$I(S_i) = 2\pi\left[2g_i - 2 + \sum_{p \in \overline{S}_i}\left(1 - \frac{1}{n(p)}\right)\right]. \tag{1}$$

The information given above is contained in [1], and it will be our starting point, as it was for Bers. In other respects, we shall strive to make the presentation self-contained.

3. Let q be any integer. If a meromorphic function ϕ on Ω satisfies $\phi(Az)A'(z)^q = \phi(z)$ for all $A \in \Gamma$, it determines through projection a differential $\tilde{\phi}\,d\tilde{z}^q$ on S.

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Definition 1. We call ϕ a *Γ-differential of order* q if ϕ can be extended to a meromorphic differential on $\bar{S} = \bigcup \bar{S}_i$.

The linear space of such differentials will be denoted by $D^q = D^q(\Gamma)$. It is a free sum of the spaces D_i^q formed by differentials that vanish identically outside of $\Omega_i = \pi^{-1}(S_i)$.

We shall also use $\tilde{D}^q$ and $\tilde{D}_i^q$ for the spaces of differentials on $\bar{S}$ and $\bar{S}_i$.

4. At $p \in S$, let $\tilde{\nu}$ be the degree (order) of $\tilde{\phi}$, and ν the degree of ϕ at points $z \in \pi^{-1}(p)$. As above, we assume that the projection is given locally by $\tilde{z} = z^n$, $n = n(p)$. From $\tilde{\phi} d\tilde{z}^q = \phi dz^q$ we conclude that $n(\tilde{\nu} + q) = \nu + q$. The possible values of ν differ by multiples of n, and it is readily seen that $\nu \geq 0$ if and only if $\tilde{\nu} \geq \tilde{\nu}_0 = -[q(1 - 1/n)]$, where $[x]$ is the greatest integer that does not exceed x. We shall say that ϕ is *regular* over p if $\tilde{\nu} \geq \tilde{\nu}_0$.

When $n(p) = \infty$, this convention must be modified, for ν is not defined. We agree that ϕ shall be considered regular if it is bounded on the diameter of the disk Δ, in other words, as $z \to 0$ through positive values. From $\tilde{z} = e^{-1/z}$ and $\tilde{\phi} d\tilde{z}^q = \phi dz^q$ we obtain $\tilde{\phi}\tilde{z}^q = \phi z^{2q}$. Hence, if $q > 0$, the boundedness of ϕ implies that $\phi z^{2q} \to 0$ and consequently $\tilde{\nu} \geq 1 - q$. In contrast, if $q \leq 0$, we may only deduce that $\tilde{\nu} \geq -q$. Accordingly, we set $\tilde{\nu}_0 = 1 - q$ if $q > 0$ and $\tilde{\nu}_0 = -q$ if $q \leq 0$, and we say that ϕ is *regular over* p if $\tilde{\nu} \geq \tilde{\nu}_0$.

With reference to the same normalization as before, we record the following facts.

LEMMA 1. *If* $q > 0$, *ϕ is regular over* p *whenever* $\phi = o(|z|^{-2q})$ *as* $z \to 0$ *through positive values, and this implies*

$$\phi = O(|z|^{-2q} e^{-1/|z|}).$$

If $q \leq 0$, *ϕ is regular whenever* $\phi = o(|z|^{-2q} e^{1/|z|})$, *and this implies* $\phi = O(|z|^{-2q})$.

The proof is trivial.

3. To summarize our definition of regularity, it is expedient to write

$$m^q(p) = \begin{cases} [q(1 - 1/n(p))] & \text{if } n(p) < \infty, \\ q - 1 & \text{if } n(p) = \infty,\ q > 0, \\ q & \text{if } n(p) = \infty,\ q \leq 0. \end{cases} \tag{2}$$

It is of some interest to observe that the last two cases are limiting cases of the first: $m^q(p) = \lim_{n \to \infty} [q(1 - 1/n)]$.

We define the *ramification divisor of order* q as the divisor α^q on $\bar{S}$ with coefficients $-m^q(p)$. According to Section 4, ϕ is regular if ϕ is a multiple of α^q.

More generally, let α be any divisor on S with zero coefficients at the branch points. We say that ϕ is a *multiple of* α if ϕ is a multiple of $\alpha^q + \alpha$. The space of such multiples is denoted by $D^q(\alpha)$, and its restriction to Ω by $D_i^q(\alpha)$. Similarly, $\tilde{D}^q(\beta)$ and $\tilde{D}_i^q(\beta)$ refer to multiples of β on $\bar{S}$ and $\bar{S}_i$. Obviously,

(3) $$D_i^q(\alpha) \simeq \tilde{D}_i^q(\alpha^q + \alpha).$$

6. The divisors α^q and α^{1-q} are closely related.

LEMMA 2. $\alpha^q = -\alpha^{1-q}$.

For $n(p) = \infty$, it follows trivially from (2) that $m^q = -m^{1-q}$. For $n(p) = n < \infty$, we find that

$$m^q + m^{1-q} \leq q(1 - 1/n) + (1 - q)(1 - 1/n) = 1 - 1/n,$$

and for diophantine reasons,

$$m^q + m^{1-q} \geq q(1 - 1/n) - (1 - 1/n) + (1 - q)(1 - 1/n) - (1 - 1/n) = -1 + 1/n.$$

Hence $m^q + m^{1-q} = 0$, and the lemma follows.

7. The Riemann-Roch theorem gives a relation between the dimensions of $\tilde{D}_i^q(\alpha^q + \alpha)$ and $\tilde{D}_i^{1-q}(\alpha^{1-q} - \alpha)$, that is, according to (3), between the dimensions of $D_i^q(\alpha)$ and $D_i^{1-q}(-\alpha)$.

In the following, α_i denotes the restriction of α to $\overline{S}_i$.

LEMMA 3.

(4) $$\dim D_i^q(\alpha) = \dim D_i^{1-q}(-\alpha) + \sum_{\overline{S}_i} m^q(p) - \deg \alpha_i + (2q - 1)(g_i - 1).$$

We omit the standard proof, but remark that $D_i^q(\alpha) = 0$ whenever

$$\deg(\alpha_i^q + \alpha_i) > 2q(g_i - 1).$$

This is true when $\deg \alpha_i$ is sufficiently large. We shall need the following special case:

LEMMA 4. $D_i^q(\alpha) = 0$ *if* $\deg \alpha_i \geq 0$ *and* $q < 0$.

Indeed, by virtue of (1),

$$\deg(\alpha_i^q + \alpha_i) \geq -\sum_{\overline{S}_i} m^q(p) \geq -\sum_{\overline{S}_i} q(1 - 1/n) > 2q(g_i - 1).$$

In particular, there are no everywhere regular differentials with $q < 0$.

For $q \geq 2$, we obtain from Lemmas 3 and 4

(5) $$\dim D_i^q(0) = \sum_{p \in S} m^q(p) + (2q - 1)(g_i - 1),$$

and it follows further that (4) can be rewritten as

(6) $$\dim D_i^q(\alpha) = \dim D_i^{1-q}(-\alpha) + \dim D_i^q(0) - \deg \alpha.$$

8. Throughout the remaining part of the paper, we assume $q \geq 2$.

Definition 2. A meromorphic function f on Ω is called an *Eichler integral of order* $1 - q$ if $f^{(2q-1)} \in D^q$.

We denote the space of Eichler integrals by E^{1-q}. The study of Eichler integrals is based on the following identity, discovered by G. Bol and quoted in [3].

LEMMA 5.

$$D^{2q-1}[f(Az)A'(z)^{1-q}] = f^{(2q-1)}(Az)A'(z)^{q}. \tag{7}$$

Here A is an arbitrary linear transformation, and f needs to be defined and analytic only in a neighborhood of a point $Az_0 \neq \infty$. Let γ be a small circle about Az_0. By Cauchy's integral formula and the identity

$$(A\zeta - Az)^2 = (\zeta - z)^2 A'(\zeta)A'(z),$$

we obtain for Az inside γ the equations

$$f^{(2q-1)}(Az) = \frac{(2q-1)!}{2\pi i}\int_{\gamma}\frac{f(\zeta)d\zeta}{(\zeta - Az)^{2q}} = \frac{(2q-1)!}{2\pi i}\int_{A^{-1}\gamma}\frac{f(A\zeta)A'(\zeta)d\zeta}{(A\zeta - Az)^{2q}}$$

$$= \frac{(2q-1)!}{2\pi i}\int_{A^{-1}\gamma}\frac{f(A\zeta)A'(\zeta)^{1-q}d\zeta}{(\zeta - z)^{2q}A'(z)^{q}} = A'(z)^{-q}D^{2q-1}[f(Az)A'(z)^{1-q}].$$

The second application of Cauchy's formula is valid if z is inside $A^{-1}\gamma$, and this is so for sufficiently small γ because $A^{-1}\infty \neq z_0$.

9. As an immediate consequence, we obtain the following.

LEMMA 6. $D^{1-q} \subset E^{1-q}$.

For another consequence, assume that $f \in E^{1-q}$, and set

$$f(Az)A'(z)^{1-q} - f(z) = P^{A}f(z).$$

Then $(P^{A}f)^{(2q-1)} = 0$, and we conclude that $P^{A}f$ is a polynomial of degree at most $2q - 2$, in each component of Ω. The polynomials $P^{A}f$ will ordinarily differ from component to component.

The $P^{A}f$ are called *periods* of f, and they are interdependent according to the scheme

$$P^{AB}f(z) = P^{A}f(Bz)B'(z)^{1-q} + P^{B}f(z). \tag{8}$$

We conclude that all periods can be computed from the periods $P^{A_i}f$, which correspond to the generators $A_1, \cdots, A_N$. In particular, $f \in D^{1-q}$ if all $P^{A_i}f$ are 0.

10. It will now be assumed that ∞ is neither a limit point nor an elliptic fixed point. Under this condition, it is easy to see that for each $q \geq 2$ the series $\sum_A |A'(\zeta)|^q$ converges uniformly on compact subsets of D.

By virtue of the convergence, the function

$$\phi(z, \zeta) = \sum_{A \in \Gamma}\frac{A'(\zeta)^{q}}{z - A\zeta} \tag{9}$$

is meromorphic in both variables, for $z, \zeta \in \Omega$. As a function of ζ, the series is a Poincaré theta-series, and it is a matter of easy verification that

$$\phi(z, A\zeta)A'(\zeta)^q = \phi(z, \zeta).$$

It can also be shown that the differential is regular except for at most a simple pole at z.

However, for our purposes it is the properties of $\phi(z, \zeta)$ as a function of z that are important. It is almost obvious that $\phi(z, \zeta) \in E^{1-q}$, but the crucial fact is that the periods are the same in all components.

LEMMA 7. *The periods* $P^A \phi(z, \zeta)$ *are equal to the same polynomial of* z, *in all of* Ω.

The proof is computational. For a fixed $B \in \Gamma$, the summation in (9) can be taken over all AB, and we obtain the equations

$$\phi(Bz, \zeta) = \sum_{A \in \Gamma} \frac{B'(A\zeta)^q A'(\zeta)^q}{Bz - BA\zeta} = B'(z)^{-1/2} \sum_A \frac{B'(A\zeta)^{q-1/2} A'(\zeta)^q}{z - A\zeta},$$

from which we deduce that

$$\phi(Bz, \zeta)B'(z)^{1-q} - \phi(z, \zeta) = \sum_A \left[\left(\frac{B'(A\zeta)}{B'(z)}\right)^{q-1/2} - 1 \right] \frac{A'(\zeta)^q}{z - A\zeta}.$$

Here, since $B'(z)$ is of the form $(cz + d)^{-2}$, one recognizes that the bracketed expression is a polynomial in z of degree $2q - 1$ and that it is divisible by $z - A\zeta$. It follows that $P^B \phi(z, \zeta)$ is indeed a polynomial of degree at most $2q - 2$.

11. To complete the study of the function $q(z, \zeta)$, we need to investigate its behavior at a parabolic puncture. With the same notations as in Section 4, the points $A\zeta$ lie on or outside a circle passing through 0, and as we approach the fixed point along a diameter, we have the inequality $|z - A\zeta| < |z|$ for sufficiently small $|z|$. It follows from (9) that the regularity condition in Lemma 1 (with $1 - q < 0$) is amply fulfilled.

Therefore, any period-free linear combination of functions $\phi(z, \zeta)$ is a differential of order $1 - q$, regular over the punctures.

12. Let α be a divisor formed by the projections of m distinct points $\zeta_1, \cdots, \zeta_m \in \Omega$ different from the elliptic fixed points.

LEMMA 8.

$$\dim D^{1-q}(-\alpha) \geq \deg \alpha - (N - 1)(2q - 1). \tag{10}$$

Consider the linear space spanned by $\phi(z, \zeta_1), \cdots, \phi(z, \zeta_m)$ together with all polynomials of degree at most $2q - 2$. This subspace of E^{1-q} has dimension $m + 2q - 1$, and in order that an element have zero periods over $A_1, \cdots, A_N$ it must satisfy $N(2q - 1)$ linear conditions. The lemma is proved.

Actually, (10) is true for arbitrary α, but the proof becomes more complicated.

13. We combine (10) and (6) to obtain

$$\sum_i (\dim D_i^q(0) - \dim D_i^q(\alpha)) \leq (N - 1)(2q - 1). \tag{11}$$

The sum needs to be extended only over the terms with $\alpha_i > 0$. But for these terms we may choose α_i so large that $\dim D_i^q(\alpha) = 0$. The result applies to an arbitrary selection of terms, and we have proved the following result.

MAIN THEOREM.

$$\dim D^q(0) \leq (N - 1)(2q - 1). \tag{12}$$

This is Bers' Corollary 2, but we prefer to call it his main theorem because it contains basic information and is stronger than the area theorem.

14. The area theorem follows when we express $\dim D^q(0) = \sum_i \dim D_i^q(0)$ by use of (5) and let $q \to \infty$. We obtain the inequality

$$\sum_i \left(\sum_{p \in \overline{S}_i} (1 - 1/n) + 2g_i - 2 \right) \leq 2(N - 1),$$

and we deduce from (1) that the total area is at most $4\pi(N - 1)$.

15. We conclude by showing that (12) leads to a better estimate of the number of components. We need to know when $D_i^q(0)$ does not reduce to 0.

First of all, if $g_i \geq 2$, it follows immediately from (5) that $\dim D_i^q \geq 2q - 1 > 0$. The same is true if $g_i = 1$, for then there is at least one puncture, and $m^q(p) \geq [q/2] > 0$.

It remains to consider the punctured spheres. We contend that in all cases $\dim D_i^4(0) + \dim D_i^6(0) > 0$. Since the dimensions do not decrease when the ramification increases, we need to check only the lowest permissible signatures, namely (2, 2, 2, 2, 2), (2, 2, 2, 3), (2, 3, 7), (2, 4, 5), and (3, 3, 4). The result is as follows:

	$\dim D^4(0)$	$\dim D^6(0)$
(2, 2, 2, 2, 2)	3	4
(2, 2, 2, 3)	1	1
(2, 3, 7)	0	1
(2, 4, 5)	1	0
(3, 3, 4)	0	1

We add the inequalities (12) for $q = 4$ and $q = 6$, and conclude that the number of components is at most $18(N - 1)$.

It would be useful to know all cases with $\dim D^q(0) = 0$. The only candidates are the punctured spheres. If the dimension is 0, we must have

$$2q - 1 = \sum [q(1 - 1/n)] \geq (q - 1) \sum (1 - 1/n) \geq (q - 1)(2 + 1/42),$$

which gives $q \leq 43$. Actually, $q = 43$ with signature (2, 3, 7) does give dimension 0. To check the other cases is too laborious, unless a real need arises.

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