WILD POINTS OF CELLULAR SUBSETS OF SPHERES IN S^3, II

L. D. Loveland

The set of wild points of a cellular finite graph on a 2-sphere in S^3 is either empty or degenerate, or it contains an arc [6, Theorem 1]. Furthermore, such a finite graph cannot contain two isolated wild points [6, Theorem 2]. The purpose of this note is to indicate how we can use a theorem recently proved by D. R. McMillan, Jr. [9, Theorem 1], together with the results of [6], to obtain similar results for arbitrary cellular subsets of 2-spheres in S^3.

A key to the proofs in [6] is the result by Burgess [4] that a 2-sphere S in S^3 has at most two wild points if each component of $S^3 - S$ is an open 3-cell and S is locally tame modulo a 0-dimensional set.

A subset X of S^3 is said to be cellular if there exists a sequence of 3-cells $C_1, C_2, \cdots$ such that $X = \bigcap_1^\infty C_i$ and $C_{i+1} \subset \text{Int}\, C_i$ for each i. (For other definitions, consult the references.)

THEOREM 1. *A cellular subset of a 2-sphere* S *in* S^3 *cannot contain two isolated wild points of* S.

Proof. Suppose X is a cellular subset of S such that X contains two isolated wild points p and q of S. Some arc A in S contains points p and q such that A is locally tame modulo $\{p, q\}$ (see [7] or [8]), and some disk D containing $X \cup A$ is locally polyhedral modulo $X \cup A$ [2]. Since A is locally tame modulo two points of X, we see that D is locally tame modulo X [5]; hence D is cellular. According to McMillan [9], this implies that A is cellular. It follows that A is locally tame either at p or at q [6], say at p. Since p is an isolated wild point of S and p lies in a tame arc in S, we have a contradiction [5].

Note. There are examples of cellular arcs that lie on a 2-sphere and contain exactly two wild points [1] (in fact, infinitely many wild points [3]) of the sphere.

THEOREM 2. *If* X *is a cellular subset of a* 2-*sphere* S *in* S^3 *such that* S *is locally tame* modulo X *and the set* W *of wild points of* S *is* 0-*dimensional, then* W *contains at most one point.*

Proof. There exists an arc A in S containing W [10], and there exists a disk D in S such that $A \cup X \subset D$. Since D is locally tame modulo X, it follows that D is cellular; hence A is also cellular [9]. Therefore A has at most one wild point [6], and it follows from [5] that W contains at most one point.

THEOREM 3. *If* X *is a cellular subset of a* 2-*sphere* S *in* S^3 *and* S *is locally tame* modulo X, *and if* W *is the set of wild points of* S, *then either* W *is empty,* W *is degenerate, or* W *contains a nondegenerate continuum.*

Furthermore, if U *is an open subset of* S *such that* $U \cap W$ *is* 0-*dimensional, then* $W \cap U$ *contains at most one point.*

Proof. Suppose every continuum in W is degenerate. Then W is totally disconnected — hence W is 0-dimensional. It follows from Theorem 2 that W contains at most one point.

Received August 29, 1967.

Suppose that U is an open subset of S, that $U \cap W$ is 0-dimensional, and that $W \cap U$ contains two points p_1 and p_2. Let U_1 and U_2 be components of U, and let D_1 and D_2 be disks such that $p_i \in \text{Int } D_i \subset D_i \subset U_i$ for $i = 1, 2$. Since $D_i \cap W$ is a closed 0-dimensional subset of U_i, there exist arcs A_1 and A_2 such that $D_i \cap W \subset A_i \subset U_i$ for $i = 1, 2$ [10]. Since a disk D on S that contains $X \cup A_1 \cup A_2$ is cellular, both A_1 and A_2 are cellular [9]. Therefore each A_i is locally tame modulo the point p_i [6], and so p_1 and p_2 are isolated wild points of S [5]. This contradicts Theorem 1, since the p_i lie in X.

Note. Let X be a continuum in the boundary S' of a cellular 3-cell such that X does not separate S'. Since X is cellular (see Theorem 4 of [7]), and since X lies on a 2-sphere S that is locally tame modulo X [2], the conclusion of Theorem 3 is satisfied relative to the set W of wild points of S.

REFERENCES

1. J. W. Alexander, *An example of a simply connected surface bounding a region which is not simply connected.* Proc. Nat. Acad. Sci. U.S.A. 10 (1924), 8-10.

2. R. H. Bing, *Approximating surfaces with polyhedral ones.* Ann. of Math. (2) 65 (1957), 456-483.

3. ———, *A wild surface each of whose arcs is tame.* Duke Math. J. 28 (1961), 1-15.

4. C. E. Burgess, *Criteria for a 2-sphere in S^3 to be tame modulo two points.* Michigan Math. J. 14 (1967), 321-330.

5. P. H. Doyle and J. G. Hocking, *Some results on tame disks and spheres in E^3.* Proc. Amer. Math. Soc. 11 (1960), 832-836.

6. L. D. Loveland, *Wild points of cellular subsets of 2-spheres in S^3.* Michigan Math. J. 14 (1967), 427-431.

7. ———, *Piercing points of crumpled cubes.* (Submitted for publication.)

8. J. Martin, *Tame arcs on disks.* Proc. Amer. Math. Soc. 16 (1965), 131-133.

9. D. R. McMillan, Jr., *Piercing a disk along a cellular set.* Proc. Amer. Math. Soc. 19 (1968), 153-157.

10. R. L. Moore and J. R. Kline, *On the most general plane closed point-set through which it is possible to pass a simple continuous arc.* Ann. of Math. (2) 20 (1918/19), 218-223.

Utah State University
Logan, Utah 84321