A UNIQUE-EMBEDDING THEOREM IN CODIMENSION 1

Edward M. Brown

1. INTRODUCTION

The problem of determining the equivalence classes of embeddings of one manifold into another has been successfully attacked in two general cases. In the first case one assumes that the difference in the dimensions of the manifolds is sufficiently large, and one shows that there exists a unique class of embeddings (see for example [5], [12], or [13]). In the other case one singles out a particular class (generally called unknotted) and reduces its determination to some homotopy problem (see [1], [6], [9], or [12]). There are scattered results of another nature, where one has a unique embedding class even for small codimension (see [2], [3], [4], [7]). This paper demonstrates yet another of the "scattered" results, namely:

THEOREM. *Let* M *be a* 2-*sphere with an odd number of crosscaps. Let* $f\colon M \to M \times [0, 1]$ *be an embedding into the interior of* $M \times [0, 1]$. *Then* f(M) *can be moved onto* $M \times \{1/2\}$ *by an ambient isotopy that leaves* $M \times \{0, 1\}$ *pointwise fixed.*

If M is any other closed connected 2-manifold, the above result is false. For example, a torus T can be embedded in $T \times [0, 1]$ as the boundary of a tube around a knot, and this can be done in infinitely many different ways.

2. DEFINITIONS AND A LEMMA

We work entirely in the combinatorial category; in fact, we work only with compact combinatorial manifolds of dimension 2 or 3, with (possibly empty) boundary. Thus an embedding f of a manifold M into a manifold N will be piecewise linear. It is said to be *proper* if it carries the interior of M into the interior of N and the boundary of M into the boundary of N (that is, if $f(\partial M) = (\partial N) \cap f(M)$).

Two embeddings f and g of M into N are said to be *ambient isotopic* if there exists a continuous family F_t $(0 \leq t \leq 1)$ of homeomorphisms of N onto itself such that F_0 is the identity map and $F_1(f(M)) = g(M)$. (Thus ambient isotopy is a condition on the images, rather than on the functions themselves.)

Recall, finally, that a compact connected 2-manifold is a sphere with a number m of holes and a number n of handles if it is orientable. If it is not orientable, it has a number m of holes and a number n of crosscaps. The pair (m, n) and the orientability specify the manifold up to homeomorphism (see, for example, [11]).

LEMMA. *Let* M *be a compact, connected* 2-*manifold with vacuous boundary. Let* $f\colon M \to M \times [0, 1]$ *be a proper embedding. Then either* f(M) *separates* $M \times \{0\}$ *from* $M \times \{1\}$ *in* $M \times I$, *or* f(M) *is the boundary of a* 3-*dimensional submanifold of* $M \times I$.

Proof. Triangulate M and $M \times [0, 1]$ so that f is a simplicial homeomorphism. Now consider the induced homomorphism $f_*\colon H_2(M; Z_2) \to H_2(M \times [0, 1]; Z_2)$. Since

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both groups are Z_2, f_* is either the zero homomorphism or the unique isomorphism. Considered as cycles, M and $M \times \{0\}$ generate the respective groups. Thus the two cases are that f(M) is homologous to zero and that f(M) is homologous to $M \times \{0\}$, respectively. In the former case, there exists a 3-chain with boundary f(M). The 3-simplexes of this chain make up a 3-manifold whose manifold boundary is f(M). In the latter case, there exists a 3-chain with boundary $f(M) - (M \times \{0\})$, and the 3-simplexes of this chain make up a 3-manifold B with manifold boundary $\partial B = f(M) \cup (M \times \{0\})$. The topological boundary of B in $M \times [0, 1]$ is then f(M); therefore f(M) separates $M \times [0, 1]$ into B and $\overline{(M \times [0, 1]) - B}$. We need only show that $M \times \{1\}$ is contained in $\overline{(M \times [0, 1]) - B}$; but this is clear, since otherwise $M \times \{1\}$ would be part of the manifold boundary ∂B of B.

3. PROOF OF THE THEOREM

Assume now that M is a sphere with an odd number of crosscaps and an empty boundary, and that f is a proper embedding of M into $M \times [0, 1]$. Thus, by the lemma, the set f(M) either separates $M \times \{0\}$ from $M \times \{1\}$ or it is the boundary of a 3-dimensional submanifold of $M \times [0, 1]$. We show that the latter case does not arise, by showing that f(M) is not the boundary of any 3-manifold.

According to a theorem of Pontrjagin [10], if M is the boundary of any 3-manifold, then all the Stiefel-Whitney numbers of M are zero. (For a definition of the Stiefel-Whitney numbers and an elegant proof of Pontrjagin's theorem, see Milnor [8, pp. 16-19].) Thus the projective plane P^2 does not bound any 3-manifold, since the Stiefel-Whitney number corresponding to the second Stiefel class W_2 is 1. If M is a sphere with $2n + 1$ crosscaps, then there exists a 3-manifold B whose boundary is the disjoint union $P^2 \cup M$. To see this, let B_1 be a 3-manifold bounded by a sphere with 2n crosscaps (for example, let B_1 be the cartesian product of [0, 1] with a sphere having n crosscaps and one hole). Attach B_1 to $P^2 \times [0, 1]$ by identifying a disk on the boundary of B_1 with a disk on $P^2 \times \{1\}$. This produces B. If M bounded a manifold C, then attaching C to B along M would give a manifold bounded by P^2. We conclude that f(M) cannot be the boundary of a 3-dimensional submanifold of $M \times [0, 1]$, and hence that it separates $M \times \{0\}$ from $M \times \{1\}$.

We now produce the desired isotopy. Choose ε $(0 < \varepsilon < 1/2)$ so small that f(M) lies between $M \times \{\varepsilon\}$ and $M \times \{1\}$. Since f(M) separates these sets in $M \times [\varepsilon, 1]$, there exists (by [2]) a homeomorphism H of $M \times [\varepsilon, 1]$ onto itself that carries $M \times \{1/2\}$ onto f(M) and leaves $M \times \{\varepsilon, 1\}$ pointwise fixed. Extend H to be the identity function on $M \times [0, \varepsilon]$. Choose a continuous family G_t $(0 \leq t \leq 1/2)$ of homeomorphisms of $M \times [0, 1]$ onto itself such that

$$G_t \mid M \times \{0, 1\} = \text{identity},$$

$$G_0 = \text{identity},$$

$$G_{1/2}(M \times \{1/2\}) = M \times \{\varepsilon\},$$

and let

$$F_t = H \circ G_t \circ H^{-1} \qquad (0 \leq t \leq 1/2),$$

$$F_t = G^{-1}_{t-1/2} \circ F_{1/2} \qquad (1/2 \leq t \leq 1).$$

Then F_t is the desired ambient isotopy, and the proof is complete.

We remark that the usual definition of ambient isotopy in the combinatorial category requires the map $F: N \times I \to N \times I$, given by $F(n, t) = F_t(n)$, to be piecewise linear. The above theorem is true even with this definition of ambient isotopy; one need only choose the family G_t to be an ambient isotopy in the more restricted sense.

We end with a corollary that extends the theorem. If M is a compact connected 2-manifold with boundary, we denote by M^* the result of "completing" M to a manifold without boundary by attaching a disk to each hole in M.

COROLLARY. *Let* M *be a sphere with* m *holes and* $2k+1$ *crosscaps, and let* f *be a proper embedding of* M *into* $M \times (0, 1)$. *Then* f(M) *is ambient isotopic in* $M \times [0, 1]$ *to* $M \times \{1/2\}$ *by an isotopy that does not move points of* $M \times \{0, 1\}$. *In particular,* f *carries components of* ∂M *one-to-one into components of* $(\partial M) \times [0, 1]$.

Proof. We complete f to an embedding f^* of M^* in $M^* \times (0, 1)$; this can be done so that $f^*(M^* - M) \subset (M^* - M) \times (0, 1)$. By the lemma, $f^*(M^*)$ separates $M^* \times \{0\}$ from $M^* \times \{1\}$. Since

$$f^{*^{-1}}(M \times [0, 1]) = M = f^{-1}(M \times [0, 1]),$$

we see that f(M) separates $M \times \{0\}$ from $M \times \{1\}$ in $M \times [0, 1]$. We may now use the same isotopy as in the proof of the theorem to move f(M) to $M \times \{1/2\}$.

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Dartmouth College, Hanover, New Hampshire 03755