NORMS OF POWERS OF ABSOLUTELY CONVERGENT FOURIER SERIES IN SEVERAL VARIABLES

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In this paper we establish an upper bound for $\|f^n\|$, where f is an absolutely convergent Fourier series

$$f(\theta) = \sum_{\alpha} a_\alpha e^{i(\alpha,\theta)}$$

in k variables, with $\|f\| = \sum_\alpha |a_\alpha|$; here we use the notation $\alpha = (\alpha_1, \cdots, \alpha_k)$ for a k-tuple of integers, and we write $\theta = (\theta_1, \cdots, \theta_k)$ and $(\alpha, \theta) = \sum \alpha_j \theta_j$. We also use

$$D_j = \frac{\partial}{\partial \theta_j}, \qquad D^\beta = \prod_1^k D_j^{\beta_j}.$$

We introduce the partial ordering

$$\beta \geq \beta' \text{ if and only if } \beta_j \geq \beta_j' \text{ for } j = 1, \cdots, k.$$

Let $0 = (0, \cdots, 0)$ and $I = (1, \cdots, 1)$.

THEOREM. *Let* f *be given by an absolutely convergent Fourier series, and let* $|f(\theta)| \leq 1$ *for all* θ. *Suppose* $D^\beta f$ $(0 \leq \beta \leq I)$ *exists in the sense of Sobolev and belongs to* L_2. *Then*

$$\|f^n\| \leq M n^{k/2} \qquad (n = 1, 2, \cdots).$$

Remarks. For $k = 1$, the theorem was proved by Kahane (see [4, page 103]) by means of an inequality of F. Carlson [1]. We shall prove a generalization of Carlson's inequality (Lemma 2).

Kahane [3] showed that for $k = 1$ the estimate is the best possible estimate. His example is easily modified to show that

$$\|f^n\| \geq C n^{k/2} \qquad (C > 0,\ n = 1, 2, \cdots)$$

if $f(\theta) = e^{i\phi(\theta)}$ and ϕ is real, $\phi \in C^2$, and if for some θ the matrix $[D_h D_j\, \phi(\theta)]$ does not have zero as an eigenvalue. It is sufficient to deal with the localized problem, and we may rotate the coordinates to diagonalize the second derivatives (see [2]).

The proof is based on two lemmas. The first concerns polynomials in a complex variable $z = (z_1, \cdots, z_k)$. We use the notation $dz = dz_1 \cdots dz_k$ and $d\theta = d\theta_1 \cdots d\theta_k$.

LEMMA 1. *Let* $b_\alpha \geq 0$ $(\alpha \geq 0)$. *Suppose* $g(z) = \sum b_\alpha z^\alpha$ *is a polynomial. Then*

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$$\int_0^1 \cdots \int_0^1 g(z)\,dz \le 2^{-k} \int_0^{2\pi} \cdots \int_0^{2\pi} |g(e^{i\theta})|\,d\theta\,,$$

where $g(e^{i\theta}) = g(e^{i\theta_1}, \cdots, e^{i\theta_k})$.

Proof. The lemma follows from the elementary identities

$$\int_0^1 \cdots \int_0^1 g(z)\,dz = \sum_{\alpha \ge 0} \frac{b_\alpha}{\prod_j (\alpha_j + 1)}$$

and

$$\int_0^{2\pi} \cdots \int_0^{2\pi} g(e^{i\theta}) e^{i(\theta_1 + \cdots + \theta_k)} \prod_j (\pi - \theta_j)\,d\theta = (2\pi i)^k \sum_{\alpha \ge 0} \frac{b_\alpha}{\prod_j (\alpha_j + 1)}.$$

LEMMA 2 *(a generalized Carlson inequality). Let* $a_\alpha \ge 0$ *for* $\alpha \ge I$, *and let* $\sum_{\alpha \ge I} \alpha^{2I} a_\alpha^2 < \infty$. *Then*

$$\left(\sum_{\alpha \ge I} a_\alpha \right)^2 \le 2\pi^k \sum_{0 \le \beta \le I} \left(\sum_{\alpha \ge I} \alpha^{2\beta} a_\alpha^2 \right)^{1/2} \left(\sum_{\alpha \ge I} \alpha^{2(I-\beta)} a_\alpha^2 \right)^{1/2}.$$

Proof. Let N be a positive integer, and let

$$f_N(z) = \sum_{I \le \alpha \le NI} a_\alpha z^\alpha .$$

Then, with the notation $D_{z_j} = \frac{\partial}{\partial z_j}$ and $D_z^\beta = \prod D_{z_j}^{\beta_j}$, we have the relation

$$f_N^2(I) = \int_0^1 \cdots \int_0^1 D_z^I f_N^2(z)\,dz = 2 \sum_{0 \le \beta \le I} \int_0^1 \cdots \int_0^1 (D_z^\beta f_N)(D_z^{I-\beta} f_N)\,dz\,.$$

We now apply Lemma 1 and the Schwartz inequality to get the inequality

$$f_N^2(I) \le 2^{-(k-1)} \sum_{0 \le \beta \le I} \int_0^{2\pi} \cdots \int_0^{2\pi} |D_z^\beta f_N(e^{i\theta})|\,|D_z^{I-\beta} f_N(e^{i\theta})|\,d\theta$$

$$\le 2^{-(k-1)} \sum_{0 \le \beta \le I} \|D^\beta f_N(e^{i\theta})\|_{L_2} \|D^{I-\beta} f_N(e^{i\theta})\|_{L_2}\,.$$

It follows from this and from Parseval's theorem that

$$\left(\sum_{I \le \alpha \le NI} a_\alpha \right)^2 \le 2\pi^k \sum_{0 \le \beta \le I} \left(\sum_{\alpha \ge I} \alpha^{2\beta} a_\alpha^2 \right)^{1/2} \left(\sum_{\alpha \ge I} \alpha^{2(I-\beta)} a_\alpha^2 \right)^{1/2}$$

If we let N tend to infinity, we obtain the desired inequality.

Proof of the theorem. Let

$$F(\theta) = \sum b_\alpha e^{i(\alpha,\theta)}, \qquad \sum |b_\alpha| < \infty, \qquad F^n(\theta) = \sum b_\alpha^n e^{i(\alpha,\theta)}.$$

Then it follows from Lemma 2 that

$$\left(\sum_{\alpha \geq I} |b_\alpha^n|\right)^2 \leq 2\pi^k \sum_{0 \leq \beta \leq I} \left(\sum_{\alpha \geq I} \alpha^{2\beta} |b_\alpha^n|^2\right)^{1/2} \left(\sum_{\alpha \geq I} \alpha^{2(I-\beta)} |b_\alpha^n|^2\right)^{1/2}$$

$$\leq 2^{-(k-1)} \sum_{0 \leq \beta \leq I} \|D^\beta F^n(\theta)\|_{L_2} \|D^{I-\beta} F^n(\theta)\|_{L_2} \leq M n^k.$$

Since we get similar inequalities when we sum over $\alpha_1 \leq 0$, $\alpha_j \geq 1$ $(j = 2, 3, \cdots, k)$, and so forth, the theorem is proved.

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