COMPLETE DISTRIBUTIVITY IN CERTAIN INFINITE PERMUTATION GROUPS

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1. INTRODUCTION

An ℓ-group G is said to be *completely distributive* if the order of constructing infinite joins and intersections may be interchanged. In 1939, Lorenzen [7] proved that an abelian ℓ-group can be embedded in a large cardinal product of totally ordered groups. In 1963, Conrad, Harvey, and Holland [4] showed that an abelian ℓ-group can be realized as an ℓ-subgroup of an ℓ-group of real-valued functions. Both of these embedding theorems present an abelian ℓ-group as an ℓ-subgroup of a completely distributive ℓ-group. In 1963, Holland [6] proved that any ℓ-group can be embedded in the group of order-preserving permutations of some totally ordered set. The main purpose of this note is to show that the Holland embedding realizes any ℓ-group as an ℓ-subgroup of a completely distributive ℓ-group.

Section 3 is devoted to proving that the group P(L) of order-preserving permutations of a totally ordered set L is a completely distributive ℓ-group. It follows as a corollary that the ideal radical of P(L) is trivial. In Section 4 it is shown that the isotropy subgroups of P(L) are closed convex ℓ-subgroups. In Section 5 we answer a question raised by Conrad [3], by giving an example of an ℓ-group that has a trivial ideal radical and yet fails to be completely distributive.

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2. NOTATION AND TERMINOLOGY

For standard results and definitions concerning ℓ-groups, the reader is referred to [1] and [5]. If G is an ℓ-group, $G^+ = \{x \in G \mid x \geq 1\}$ is called the *positive cone* of G. An ℓ-group G is said to be *completely distributive* if the relation

$$\bigwedge_{i \in I} \left(\bigvee_{j \in J} g_{ij} \right) = \bigvee_{f \in J^I} \left(\bigwedge_{i \in I} g_{if(i)} \right)$$

holds whenever $\{g_{ij} \mid i \in I,\ j \in J\}$ is a subset of G for which all the indicated joins and intersections exist.

If L is a totally ordered set, P(L) denotes the collection of order-preserving permutations of L. P(L) is a group under the operation of composition of functions, and it is an ℓ-group with respect to the partial order defined by the rule

$$f \geq g \text{ if and only if } f(x) \geq g(x) \text{ for each } x \in L.$$

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For f, g $\in$ P(L), the join and intersection of f and g are given by

$$(f \vee g)(x) = f(x) \vee g(x),$$

$$(f \wedge g)(x) = f(x) \wedge g(x)$$

for each $x \in L$. For $g \in P(L)$ and $z \in L$, let

$$I_g(z) = \{x \in L \mid g^n(z) \leq x \leq g^m(z) \text{ for some pair of integers } n, m\}.$$

$I_g(z)$ is called a *positive (negative, zero) interval* of g provided $g(z) > z$ $(g(z) < z$, $g(z) = z)$. The intervals of g are pairwise disjoint, convex subsets of L. The *support* S(g) of g is the union of the positive and negative intervals of g. If $g \in P(L)$ and $I_g(z)$ is an interval of g, the function h defined by

$$h(x) = \begin{cases} g(x) & \text{if } x \in I_g(z), \\ x & \text{otherwise} \end{cases}$$

is an element of P(L). For $z \in L$, the *isotropy subgroup* of P(L) at z is defined as the group $H(z) = \{f \in P(L) \mid f(z) = z\}$. The *orbit* of z is the set

$$O(z) = \{f(z) \mid f \in P(L)\}.$$

For subsets A and B of L, $A < B$ means that $a < b$ for each pair $(a, b) \in A \times B$. If $A \subseteq B \subseteq L$, A is said to be *bounded* in B provided there exist elements $b, b' \in B$ such that $\{b\} < A < \{b'\}$.

3. THE COMPLETE DISTRIBUTIVITY OF P(L)

Weinberg [8], [9] has shown that an ℓ-group G is completely distributive if and only if for each g $(1 < g \in G)$ there exists an h $(1 < h \in G)$ such that whenever $g = \vee S$ for a subset S of G^+, then $h \leq s$ for some $s \in S$. A pair (g, h) of elements of an ℓ-group G that satisfy this condition will be called a *distributive pair.*

LEMMA 1. *If* f, g, h *are elements of an ℓ-group* G, *and if* $f \geq g > 1$ *and* (g, h) *is a distributive pair, then* (f, h) *is a distributive pair.*

Proof. Suppose $f = \vee S$, where $S \subseteq G^+$. Then $g = g \wedge f = \bigvee_{s \in S} (g \wedge s)$. Thus $h \leq s \wedge g \leq s$ for some $s \in S$.

Remark. If an element $g > 1$ of an ℓ-group G is completely join-irreducible in the sense that $g = \vee S$ $(S \subseteq G^+)$ implies that $g \in S$, then (g, g) is a distributive pair.

THEOREM 1. *For each totally ordered set* L, P(L) *is a completely distributive ℓ-group.*

Proof. Throughout the proof, $g > 1$ will be a fixed element of $P(L)^+$. We shall show that g is the first member of a distributive pair. Because of Lemma 1 and the subsequent remark, we may assume that g has exactly one positive interval $I_g(z)$ (note that each element of $P(L)^+$ exceeds such an element) and that g is not completely join-irreducible. The proof is divided into four parts.

Part A. *If the orbit* O(z) *does not intersect the open interval* (z, g(z)), *then there exists an* h $(1 < h \in P(L)^+)$ *such that* $S(h) \subseteq (z, g(z))$; *and for any such* h, (g, h) *is a distributive pair.* To see this, let $s \in P(L)$ be such that $1 < s < g$. Then there exists $y \in I_g(z)$ such that $y < s(y)$, and there is an integer n such that $g^n(z) \leq y < g^{n+1}(z)$. Now $g^n(z) \leq s\,g^n(z) \leq g^{n+1}(z)$, and since $O(z) \cap (z, g(z)) = \emptyset$, it follows that $s\,g^n(z) = g^n(z)$ or $s\,g^n(z) = g^{n+1}(z)$.

If $s\,g^n(z) = g^n(z)$, then $s\,g^{n+1}(z) = g^{n+1}(z)$, and the permutation

$$\bar{s}(x) = \begin{cases} s(x) & \text{if } x \in I_s(y), \\ x & \text{otherwise} \end{cases}$$

satisfies the conditions $\bar{s} > 1$ and $S(\bar{s}) \subseteq (g^n(z), g^{n+1}(z))$. Thus $h = g^{-n}\bar{s}g^n$ satisfies $h > 1$ and $S(h) \subseteq (z, g(z))$.

If $s\,g^n(z) = g^{n+1}(z)$, then $s\,g^m(z) = g^{m+1}(z)$ for all integers m. Since $s \neq g$, there exists an interval $(g^k(z), g^{k+1}(z))$ on which gs^{-1} is not trivial. The permutation

$$\bar{h}(x) = \begin{cases} gs^{-1}(x) & \text{if } x \in (g^k(z), g^{k+1}(z)), \\ x & \text{otherwise} \end{cases}$$

satisfies the conditions $\bar{h} > 1$ and $S(\bar{h}) \subseteq (g^k(z), g^{k+1}(z))$, so that $h = g^{-k}\bar{h}g^k$ satisfies $h > 1$ and $S(h) \subseteq (z, g(z))$.

Now, if $1 < h \in P(L)^+$ and $S(h) \subseteq (z, g(z))$, then (g, h) is a distributive pair. To see this, suppose that $g = \bigvee T$, where $T \subseteq P(L)^+$. If $t(z) = z$ for each $t \in T$, then $h^{-1}g \geq t$ for each $t \in T$, and this contradicts the assumption that $g = \bigvee T$. Therefore there exists $t \in T$ such that $t(z) = g(z)$, in which case $t \geq h$.

Because of Part A, we may assume that there exists $k \in P(L)$ such that $1 < k < g$ and $z < k(z) < g(z)$.

Part B. *If* $1 < k < g$ *and* $z < k(z) < g(z)$, *then either* (g, k) *is a distributive pair, or there exists* $f \in P(L)$ *such that* $1 < f < g$ *and* S(f) *is bounded in* $I_g(z)$. To see this, suppose that (g, k) is not a distributive pair. Then there is a set $S \subseteq P(L)^+$ such that $g = \bigvee S$ and such that $s \not\geq k$ and $k \not\geq s$ for some element $s \in S$. Thus there exist $a, b \in I_g(z)$ such that $s(a) < k(a)$ and $s(b) > k(b)$. Without loss of generality we may suppose that $a < b$. If there exists $c \in I_g(z)$ such that $c < a$ and $s(c) \geq k(c)$, then $I_{s^{-1}k}(a) \subseteq (c, b)$ and the permutation

$$f(x) = \begin{cases} s^{-1}k(x) & \text{if } x \in I_{s^{-1}k}(a), \\ x & \text{otherwise} \end{cases}$$

has the property that $1 < f < g$ and S(f) is bounded in $I_g(z)$. Therefore it may be supposed that $s(x) < k(x)$ whenever $x \in I_g(z)$ and $x < a$. Let p and m be integers such that

$$g^p(z) < a \leq g^{p+1}(z),$$

$$g^m(z) \leq b < g^{m+1}(z).$$

Let

$$r = g^{m+1-p}s^{-1}kg^{-m-1+p}$$

and

$$q = k^{-1} s.$$

The intersection $q \wedge r$ satisfies the conditions

$$(q \wedge r)(b) > b,$$

$$(q \wedge r)(a) < a,$$

$$(q \wedge r)(g^{m+1-p}(b)) < g^{m+1-p}(b).$$

Thus $I_{q\wedge r}(b) \subseteq (a, g^{m+1-p}(b))$, and the permutation

$$f(x) = \begin{cases} (q \wedge r)(x) & \text{if } x \in I_{q\wedge r}(b), \\ x & \text{otherwise} \end{cases}$$

has the properties that $1 < f < g$ and $S(f)$ is bounded in $I_g(z)$.

Because of Part B, we may suppose there exists $f \in P(L)$ such that $1 < f < g$ and $S(f)$ is bounded in $I_g(z)$.

Part C. *If there exists an element* $f \in P(L)$ *such that* $1 < f < g$ *and* $S(f)$ *is bounded in* $I_g(z)$, *then there exists* $h \in P(L)$ *such that*

i) $g > h > 1$,

ii) $S(h) \subseteq (z, g^2(z))$,

iii) $S(h) < g S(h)$.

To prove this, we may suppose that f has a single supporting interval $I_f(a)$. Let

$$g^p(z) \leq S(f) \leq g^m(z),$$

where p and m are the largest and smallest integers for which this inequality holds. Let $f_1 = g^{-p} f g^p$, and let $n = m - p$. Then $1 < f_1 < g$ and $S(f_1) \subseteq (z, g^n(z))$. If $n = 1$, then $h = f_1$ satisfies the three conditions. If $n > 1$ and there exists $b \in S(f_1)$ such that

$$z < b < g(z) \quad \text{and} \quad g^{n-1}(b) \in S(f_1),$$

let $k = g^{-n+1} f_1 g^{n-1}$ and let $h = k \wedge f_1$. Then h satisfies the three conditions. The only case left is that in which $n > 1$ and for each x satisfying

$$x \in (z, g(z)) \cap S(f_1)$$

it is known that $g^{n-1}(x) \notin S(f_1)$. In this case, let $k = g^{-n+2} f_1 g^{n-2}$ and $h = k \wedge f_1$. Then h satisfies the three conditions.

Part D. *Conclusion of the Proof.* Because of Parts A, B, and C, it may be assumed that there exists $h \in P(L)$ satisfying the three conditions of Part C. Let $w \in I_g(z)$ be such that $w < h(w)$. Since $w \in S(h)$, it follows that $g(w) \notin S(h)$. Thus, for each positive integer n, $h^{-n} g(w) = g(w) > w$ and therefore $g(w) > h^n(w) > w$.

If h is the first member of a distributive pair, then so is g, because of Lemma 1. If not, then Parts A, B, and C prove the existence of $g_1 \in P(L)$ such that $h > g_1 > 1$ and $S(g_1) \subseteq (w, h^2(w))$. *In this case,* (g, g_1) *is a distributive pair.* To see this, let $g_2 = h^2 g_1 h^{-2}$. Then g_1 and g_2 satisfy the conditions

$$1 < g_1 < g, \quad 1 < g_2 < g, \quad S(g_1), S(g_2) \subseteq (w, g(w)), \quad S(g_1) < S(g_2).$$

Now suppose that $g = \vee T$, where $T \subseteq P(L)^+$. If $t(w) < S(g_2)$ for each $t \in T$, then $g_2^{-1} g \geq t$ for each $t \in T$, and this contradicts $g = \vee T$. Therefore there exist $t \in T$ and $c \in S(g_2)$ such that $c \leq t(w)$, and therefore $t \geq g_1$.

Holland [6] has shown that each ℓ-group G can be embedded as an ℓ-subgroup of P(L), for some totally ordered set L. Because of this, Theorem 1 has the following corollary.

COROLLARY 1. *Each ℓ-group can be embedded as an ℓ-subgroup in a completely distributive ℓ-group.*

This corollary suggests the following questions: *If* G *is an ℓ-group, does there exist a minimal completely distributive ℓ-group containing* G? *If two such groups exist, are they ℓ-isomorphic?* To the author's knowledge, this problem has not yet been attacked.

In [3], Conrad has shown that a completely distributive ℓ-group has a trivial ideal radical. Because of this result, the following corollary is immediate.

COROLLARY 2. *For any totally ordered set* L, *the ideal radical of* P(L) *is trivial.*

This corollary seems to indicate that the ℓ-ideals of P(L) are rather scarce. *Another open problem is that of determining the ℓ-ideal structure of* P(L). Holland's embedding theorem gives this question considerable importance.

Remark. The proof of Theorem 1 shows that every convex ℓ-subgroup of P(L) is completely distributive. This proof also shows that an ℓ-subgroup of P(L) that is *full* in the sense defined by Cohn [2] is completely distributive.

4. THE ISOTROPY SUBGROUPS OF P(L) ARE CLOSED

A convex ℓ-subgroup H of an ℓ-group G is said to be *closed* if whenever S is a subset of H such that $g = \vee S$ exists, then $g \in H$.

THEOREM 2. *For any ordered set* L *and for any element* z *of* L, *the isotropy subgroup* H(z) *is a closed convex ℓ-subgroup of* P(L).

Proof. H(z) is clearly a convex ℓ-subgroup of P(L). In order to show that H(z) is closed, it suffices to show that if $k = \vee S$, where $S \subseteq H(z)^+$, then $k \in H(z)$. Suppose then that $k \notin H(z)$, and define g by

$$g(x) = \begin{cases} k(x) & \text{if } x \in I_k(z), \\ x & \text{otherwise}. \end{cases}$$

Then $1 < g \leq k$ and $g = \bigvee_{s \in S} (g \wedge s)$. Note that $z < g(z)$ and that

$$g \wedge s \in H(z) \quad \text{for each } s \in S.$$

Thus $g \neq g \wedge s$ for each $s \in S$. If $O(z) \cap (z, g(z)) = \emptyset$, Part A of the proof of Theorem 1 demonstrates the existence of an element $h > 1$ such that $S(h) \subseteq (z, g(z))$. In this case, it is easy to see that gh^{-1} exceeds $g \wedge s$ for each $s \in S$, and this contradicts the relation $g = \bigvee_{s \in S} (g \wedge s)$. Therefore it may be assumed, as in Part B of the proof of Theorem 1, that there exists $k \in P(L)$ such that $1 < k < g$ and $z < k(z) < g(z)$. Now there exists $s \in S$ such that $g \wedge s \not\leq k$ and $g \wedge s \not\geq k$. The proofs of Parts B and C of Theorem 1 guarantee the existence of $h \in P(L)$ such that

$$\text{i) } g > h > 1,$$
$$\text{ii) } S(h) \subseteq (z, g^2(z)),$$
$$\text{iii) } S(h) < gS(h).$$

Since $g > h^{-1}g$, there exists $t \in S$ such that $g \wedge t \not\leq h^{-1}g$. Therefore there exists $w \in I_g(z)$ such that

$$g^{-1}h(g \wedge t)(w) > w.$$

If $w \notin g^{-1}S(h)$, then $h^{-1}g(w) = g(w) \geq (g \wedge t)(w)$, and this contradicts the above inequality. Thus

$$w \in g^{-1}S(h) \subseteq (g^{-1}(z), g(z)).$$

Suppose that $w \leq z$. Then

$$h(g \wedge t)(w) \leq h(g \wedge t)(z) = z \leq g(w),$$

and this again contradicts the choice of w. Thus $w \in (z, g(z))$. Now

$$g^{-1}h(g \wedge t)(z) = g^{-1}(z) < z$$

and

$$g^{-1}h(g \wedge t)(g(z)) \leq g^{-1}hg^2(z) = g(z).$$

It follows that

$$I_{g^{-1}h(g\wedge t)}(w) \subseteq (z, g(z)).$$

Define h_1 by

$$h_1(x) = \begin{cases} g^{-1}h(g \wedge t)(x) & \text{if } x \in I_{g^{-1}h(g\wedge t)}(w), \\ x & \text{otherwise}. \end{cases}$$

Then $h_1 > 1$ and $S(h_1) \subseteq (z, g(z))$. It is easy to show that $h_1^{-1}g$ exceeds $g \wedge s$ for each $s \in S$, and this contradicts $g = \bigvee_{s \in S} (g \wedge s)$. It follows that the isotropy subgroup $H(z)$ is closed.

5. AN EXAMPLE

An *ℓ-ideal* of an ℓ-group G is a normal convex ℓ-subgroup of G. The *ideal radical* L(G) of G is defined as follows: For $1 \neq g \in G$, let L_g denote the subgroup of G generated by the collection of all ℓ-ideals of G not containing g. Then $L(G) = \bigcap L_g$ $(g \in G, g \neq 1)$. Conrad [3] has shown that for a representable ℓ-group G, $L(G) = \{1\}$ if and only if G is completely distributive, and he asks if this is true for an arbitrary ℓ-group. The following example shows that the two conditions are not equivalent.

Example. Let R denote the collection of real numbers, and let $f \in P(R)$ be defined by $f(x) = x + 1$. Let

$$G = \{g \in P(R) \mid gf^m = f^m g \text{ for some positive integer } m\}.$$

Then G is an ℓ-subgroup of P(R). It will be shown that G is not completely distributive and that $L(G) = \{1\}$. In fact, G has no proper ℓ-ideals.

G *is not completely distributive.* This will be demonstrated by showing that G does not satisfy Weinberg's condition; in particular it will be shown that f is not the first member of a distributive pair. Let $h \in G$, where $1 < h$. Since h commutes with some positive power of f, S(h) is cofinal in R. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $x_n \in S(h)$ and $n + 2 \leq x_n$. For each n, let t_n be an integer such that $x_n \leq t_n$. Define g_n' on $[0, t_n]$ by

$$g_n'(x) = \begin{cases} (2^n + 1)x & \left(0 \leq x < \frac{1}{2^n}\right), \\ x + 1 & \left(\frac{1}{2^n} \leq x < n\right), \\ \frac{x}{2} + \frac{n}{2} + 1 & (n \leq x < n + 2), \\ x & (n + 2 \leq x \leq t_n). \end{cases}$$

For each n, g_n' has an extension to $g_n \in G$ satisfying $g_n f^{t_n} = f^{t_n} g_n$. Also, $f = \bigvee_{n=1}^{+\infty} g_n$, and no g_n exceeds h, since $g_n(x_n) = x_n$ and $h(x_n) > x_n$.

G *has no proper ℓ-ideals.* Suppose $N \neq \{1\}$ is an ℓ-ideal of G, and let $1 < h \in N$. Let m be the smallest positive integer such that $hf^m = f^m h$. Let [a, b] be an interval such that $[a, b] \subseteq S(h)$ and $b < a + m$. Let t be a real number $(0 < t < b - a)$ and k a positive integer such that $kt > m$. For each integer i $(0 \leq i \leq k)$, define h_i and f_i by

$$h_i(x) = h(x - it) + it, \qquad f_i(x) = x + it,$$

for each $x \in R$. Then, for each i, $h_i = f_i h f_i^{-1}$ and $h_i f^m = f^m h_i$. Also, $[a + it, b + it] \subseteq S(h_i)$. Let $g = \bigvee_{i=0}^{k} h_i$. Then $g \in N$, $gf^m = f^m g$, and $[a, a + m] \subseteq S(g)$. It follows that g has no fixed point. Therefore there is a positive number ε such that $g(x) > x + \varepsilon$ for each $x \in R$. Thus $g > g_1 > 1$, where $g_1 \in G$ is given by $g_1(x) = x + \varepsilon$. The convex ℓ-subgroup of G generated by g_1 is all of G, and therefore $N = G$. Since G has no proper ℓ-ideals, it is clear from the definition that $L(G) = \{1\}$.

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