ON THE LOEWNER DIFFERENTIAL EQUATION

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1. INTRODUCTION AND SUMMARY

1.1. Let D denote the unit disk in the z-plane. For $-\infty < \alpha < \beta < \infty$, we denote by $\mathscr{S}(\alpha, \beta)$ the class of all functions

$$f(z, t) = e^t z + a_2(t) z^2 + \cdots \qquad (z \in D,\ t \in [a, b])$$

that are analytic and univalent in D for each $t \in [\alpha, \beta]$ and satisfy the inclusion relation

$$G(s) \subset G(t) \equiv f(D, t) \equiv \{f(z, t) \colon z \in D\}$$

whenever $\alpha \leq s \leq t \leq \beta$. In a similar manner, we define $\mathscr{S}(\alpha, \infty)$ by taking $[\alpha, \infty)$ instead of $[\alpha, \beta]$.

In the terminology of [7], $\mathscr{S}(\alpha, \beta)$ is the class of all univalent normalised subordination chains over $[\alpha, \beta]$. The inclusion requirement implies that f(z, s) is univalently subordinate to f(z, t) if $\alpha \leq s \leq t \leq \beta$. This means that there exists a function

$$\phi(z, s, t) = e^{s-t} z + \cdots,$$

analytic and univalent in $|z| < 1$, such that

$$f(z, s) = f(\phi(z, s, t), t) \qquad (s \leq t). \tag{1.1}$$

It follows that

$$\phi(z, s, \tau) = \phi(\phi(z, s, t), t, \tau) \qquad (s \leq t \leq \tau). \tag{1.2}$$

The function $\phi(z, s, t)$ is absolutely continuous in s and t, and $\phi(z, s, s) = z$. Also (see [7, p. 165]),

$$f(z, s) = \lim_{t \to \infty} e^t \phi(z, s, t). \tag{1.3}$$

The class $\mathscr{S}(\alpha, \beta)$ can be characterised by a differential equation that was first considered by Loewner [6] and later by Kufarev [4].

Theorem A (see for instance [7, Satz 4]). Let $f(z, t) = e^t z + \cdots$ be analytic and univalent in $|z| < 1$ for each $t \in I = [\alpha, \beta]$. Then $f(z, t) \in \mathscr{S}(\alpha, \beta)$ if and only if

(i) the function f(z, t) is absolutely continuous on I, locally uniformly in D;

(ii) for almost all $t \in I$,

$$\frac{\partial}{\partial t} f(z, t) = z f'(z, t) h(z, t) \qquad (z \in D), \tag{1.4}$$

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where $h(z, t) = 1 + \cdots$ is analytic in $|z| < 1$ and measurable in $t \in I$, and where

$$\Re\, h(z, t) > 0 \qquad (z \in D,\ t \in I).$$

Here ' denotes the derivative with respect to z. The corresponding differential equations for the function ϕ are

$$(1.5)\qquad \begin{cases} \dfrac{\partial}{\partial s}\,\phi(z, s, t) = z\,\phi'(z, s, t)\,h(z, s), \\[2ex] \dfrac{\partial}{\partial t}\,\phi(z, s, t) = -\,\phi(z, s, t)\,h(\phi(z, s, t), t). \end{cases}$$

Every univalent function can be generated by a subordination chain f(z, t):

Theorem B [7, Folgerung 1], [2]. Let $f(z) = e^{\alpha} z + \cdots$ be analytic and univalent in D. Then there exists $f(z, t) \in \mathcal{S}(\alpha, \infty)$ such that $f(z, \alpha) = f(z)$. If $|f(z)| < e^{\beta}$, one can assume that $f(z, t) \in \mathcal{S}(\alpha, \beta)$.

1.2. Loewner's well-known slit-mapping theorem [6] asserts that if $G(\alpha)$ is a disk slit along a Jordan arc, and $G(t)$ $(\alpha \leq t \leq \beta)$ is obtained by continuously reducing this slit to a point, then f(z, t) satisfies (1.4), with

$$h(z, t) = \frac{1 + \overline{\zeta(t)}z}{1 - \overline{\zeta(t)}z} \qquad (\zeta(t) \text{ continuous in } [\alpha, \beta],\ |\zeta(t)| = 1).$$

We shall give a necessary and sufficient geometric condition that a subordination chain satisfies this particular form of the differential equation (1.4). We denote by diam E the diameter of E in the euclidian metric, and by sph dm E the diameter in the spherical metric.

THEOREM 1. *Let* $f(z, t) \in \mathcal{S}(\alpha, \beta)$ $(-\infty < \alpha < \beta < \infty)$ *and* $G(t) = \{f(z, t): z \in D\}$. *Then the following two conditions are equivalent:*

(a) *For all* $t \in [\alpha, \beta]$,

$$(1.6)\qquad \frac{\partial}{\partial t}\, f(z, t) = z f'(z, t)\, \frac{\zeta(t) + z}{\zeta(t) - z} \qquad (z \in D),$$

where the function $\zeta(t)$ *is continuous in* $[\alpha, \beta]$ *and* $|\zeta(t)| = 1$.

(b) *For every* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that whenever* $s, t \in [\alpha, \beta]$ *and* $0 \leq t - s \leq \delta$, *some cross-cut* C *of* G(t) *with* sph dm $C < \varepsilon$ *separates* 0 *from* $G(t) \setminus G(s)$.

The proof that (b) implies (a) follows the same lines as Hayman's proof [3, Chapter 6] of Loewner's slit-mapping theorem. The proof will show that it is sufficient to assume that (1.6) holds for almost all $t \in [\alpha, \beta]$.

COROLLARY. *Let* $f(z, t) \in \mathcal{S}(\alpha, \infty)$. *Suppose that* G(t) $(\alpha \leq t < \infty)$ *is the component containing* 0 *of the complement of* $B(t) = \{b(\tau): \tau \geq t\}$, *where* b(t) *is a complex-valued function continuous in* $[\alpha, \infty)$. *Then* (a) *holds.*

The corollary contains Loewner's slit-mapping theorem as a particular case. Each of our figures represents a curve B(s); in each case, the heavily drawn portion represents B(t). It should be noticed that Figure 1 gives an example for the corollary, whereas Figure 2 does not. The reason is that G(t) is strictly increasing with t,

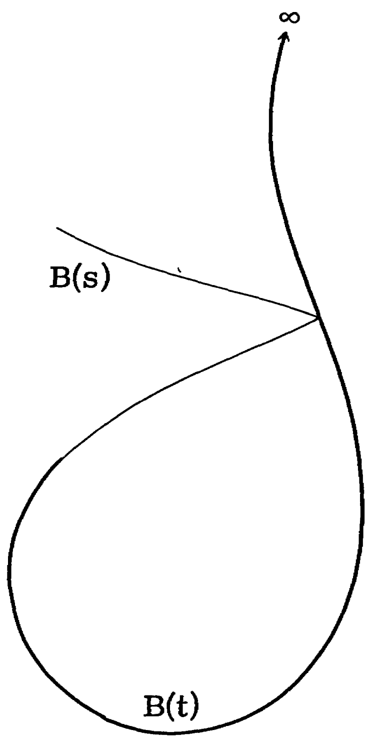

Figure 1. Example for the corollary.

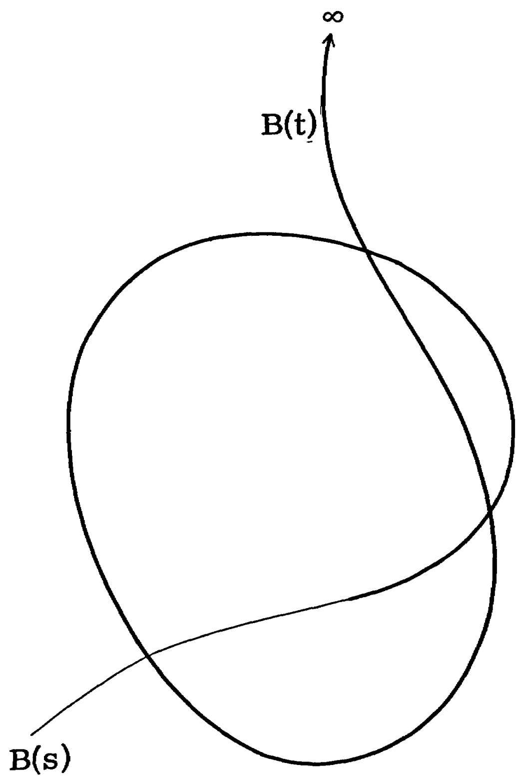

Figure 2. Neither the assumption of the corollary nor (b) is satisfied.

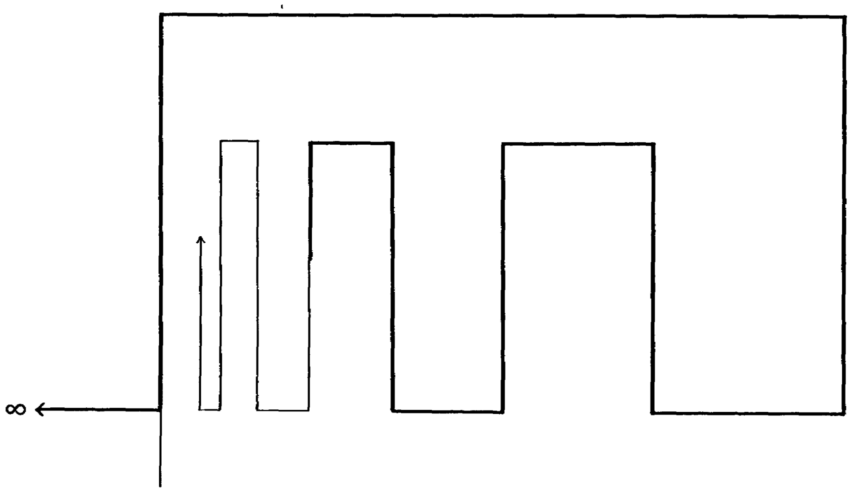

Figure 3.

because of the normalisation. In the example of Figure 3, the assumption of the corollary is not satisfied, but the conclusion (a) still holds.

1.3. It is an interesting question whether every univalent function $f(z) = e^{\alpha} z + \cdots$ can be imbedded in a subordination chain $f(z, t) \in \mathscr{S}(\alpha, \infty)$ such that $f(z, \alpha) = f(z)$ and (1.6) is satisfied almost everywhere, where $\zeta(t)$ is a *measurable* function with $|\zeta(t)| = 1$. We can only prove a weaker result, which will be deduced from Loewner's slit-mapping theorem.

THEOREM 2. *Let* $f(z) = e^{\alpha} z + \cdots$ *be analytic and univalent in* D, *and let the complement* E *of the image domain be arcwise connected on the sphere. Then there exists an* $f(z, t) \in \mathscr{S}(\alpha, \infty)$ *such that* $f(z, \alpha) = f(z)$ *and*

$$\frac{\partial}{\partial t} f(z, t) = zf'(z, t)\frac{\zeta(t)+z}{\zeta(t)-z} \qquad (z \in D), \tag{1.7}$$

where $|\zeta(t)| = 1$ *and* $\zeta(t)$ *is continuous in* (α, ∞), *except possibly on a countable set whose only limit point is* α.

Though the complement E with respect to the sphere is always connected, it need not be arcwise connected. We shall give an example where E is not arcwise connected but the conclusion of Theorem 2 still holds. In the example, E will be an indecomposable continuum that was first described by Knaster (see for instance [5, p. 143]).

2. THE CONTINUOUS CASE

2.1. We shall need some known results. The first two are proved by length-area estimates similar to [3, Theorem 2.1].

LEMMA A. *Let* g(z) *be analytic and univalent in* D. *For every* ζ $(|\zeta| = 1)$ *and* $\rho_0 > 0$, *there is a* ρ *with* $\rho_0 < \rho < 1$ *such that* $C = \{g(z): |z - \zeta| = \rho, z \in D\}$ *is a cross-cut and*

$$\text{sph dm } C \leq K_1\left(\log\frac{1}{\rho_0}\right)^{-1/2},$$

where K_1 *is an absolute constant.*

LEMMA B. *Let* g(z) *be analytic and univalent in* D, *and let* E *be a connected subset of* D. *Then*

$$\text{sph dm } g(E) \geq \exp[-K_2/(\text{diam } E)^2],$$

where K_2 *is an absolute constant.*

LEMMA C [3, Lemma 6.6]. *Suppose that* $\phi_n(z) = \gamma_n z + \cdots$ $(\gamma_n > 0)$ *is analytic and univalent in* D, *and let* $\phi_n(D) = D \setminus A_n$, *where* diam $A_n \to 0$. *Then*

$$\phi_n(z) \to z \qquad \textit{uniformly in } D.$$

If z_n $(|z_n| = 1)$ *corresponds to a point in* A_n, *then*

$$\frac{z - \phi_n(z)}{1 - \gamma_n} \sim z\frac{z_n + z}{z_n - z} \qquad (n \to \infty)$$

locally uniformly in D.

2.2. *Proof of Theorem* 1: (a) ⇒ (b).

i) Let (1.6) be satisfied for almost all $t \in [\alpha, \beta]$, and let $\eta > 0$. Since $\zeta(t)$ is a continuous function, we can choose a positive $\delta < \eta^2/16$ such that

$$|\zeta(t) - \zeta(s)| < \eta/4 \qquad (\alpha \leq s \leq t \leq \beta,\ t - s \leq \delta). \tag{2.1}$$

Let $|z| < 1$, $|z - \zeta(s)| > \eta$, $\alpha \leq s < \beta$. We want to show that

$$u(t) = |\zeta(s) - \phi(z, s, t)| > \eta/2 \qquad (s \leq t \leq \min[s + \delta, \beta]). \tag{2.2}$$

Suppose this is false. Since $u(t)$ is continuous and $u(s) = |z - \zeta(s)| > \eta$, there exists a first t_1 with $s < t_1 < s + \delta$ such that $u(t_1) = \eta/2$. Hence $u(t) \geq \eta/2$ for $s \leq t \leq t_1$, and by (2.1)

$$|\zeta(t) - \phi(z, s, t)| > \eta/4 \qquad (s \leq t \leq t_1). \tag{2.3}$$

It follows from (1.5) and (1.6) that for almost all $t \in [s, t_1]$,

$$\frac{\partial}{\partial t} u(t) = \frac{\partial}{\partial t} |\zeta(s) - \phi(z, s, t)| \geq - |\phi(z, s, t)| \left|\frac{\zeta(t) + \phi}{\zeta(t) - \phi}\right|.$$

Hence (2.3) implies that $\frac{\partial}{\partial t} u(t) \geq - 8/\eta$ for almost all $t \in [s, t_1]$. Since $u(t)$ is absolutely continuous, this implies that

$$u(t_1) - u(s) \geq -\frac{8}{\eta}(t_1 - s) \geq -\frac{8\delta}{\eta}.$$

Because $u(t_1) = \eta/2$ and $u(s) > \eta$, we find that $\delta > \eta^2/16$, contrary to our choice. Hence (2.3) holds, and therefore, by (2.1),

$$|\zeta(t) - \phi(z, s, t)| > \eta/4 \qquad (|z - \zeta(s)| > \eta,\ s \leq t \leq \min[s + \delta, \beta]). \tag{2.4}$$

ii) Let $|z - \zeta(s)| > \eta$. We want to show that

$$|\phi(z, s, t)| > |z|^2 \qquad (s \leq t \leq \min[s + \delta, \beta]). \tag{2.5}$$

Suppose this is false. Then there exists a first t_2 with $s < t_2 \leq s + \delta$ such that $|\phi(z, s, t_2)| = |z|^2$, and

$$|\phi(z, s, t)| \geq |z|^2 \qquad \text{for } s \leq t \leq t_2.$$

It follows from (1.5) and (1.6) that, for almost all $t \in [s, t_2]$,

$$\frac{\partial}{\partial t} \log |\phi(z, s, t)| = \Re\left[\frac{1}{\phi}\frac{\partial}{\partial t}\phi\right] = -\Re\left[\frac{\zeta(t) + \phi}{\zeta(t) - \phi}\right] = -\frac{1 - |\phi|^2}{|\zeta(t) - \phi|^2}.$$

Hence (2.4) implies that, for almost all $t \in [s, t_2]$,

$$\frac{\partial}{\partial t} \log |\phi(z, s, t)| \geq -\frac{16}{\eta^2}(1 - |z|^4) \geq \frac{16}{\eta^2} \log |z|.$$

Consequently

$$\log |\phi(z, s, t_2)| \geq \log |z| + \frac{16\delta}{\eta^2} \log |z|.$$

Since $\delta < \eta^2/16$, this contradicts the relation $\log |\phi(z, s, t_2)| = 2 \log |z|$.

iii) For $\alpha \leq s \leq t \leq \beta$, let

$$A(s, t) = D \setminus \phi(D, s, t). \tag{2.6}$$

Then, by (1.1),

$$G(t) \setminus G(s) = f(A(s, t), t). \tag{2.7}$$

Letting $|z| \to 1$ in (2.5), we see that

$$A(s, t) \subset \{z \in D: |z - \zeta(s)| \leq \eta\} \quad (s \leq t \leq s + \delta). \tag{2.8}$$

Corresponding to any $\varepsilon > 0$, we choose η such that

$$2K_1/(\log \eta)^{1/2} = \varepsilon .$$

Let $0 \leq t - s \leq \delta$. It follows from Lemma A that there is a ρ with $\eta < \rho < 1$ such that

$$\text{sph dm } C < \varepsilon \quad (C = f(Q, t),\ Q = \{z \in D: |z - \zeta(s)| = \rho\}).$$

By (2.8), the cross-cut Q of D separates 0 from $A(s, t)$. Hence it follows from (2.7) that the cross-cut C of $G(t)$ separates 0 from $G(t) \setminus G(s)$.

2.3. *Proof of Theorem* 1: (b) $\Rightarrow$ (a).

i) As in (2.6), we put $A(s, t) = D \setminus \phi(D, s, t)$ for $s \leq t$. By (1.2)

$$D \setminus A(s, \tau) = \phi(D \setminus A(s, t), t, \tau) \quad (s \leq t \leq \tau). \tag{2.9}$$

It follows that

$$A(s, t) \supset A(s', t) \quad (s \leq s' \leq t). \tag{2.10}$$

Let (b) be satisfied, and suppose that $t_n \geq s_n$ and $t_n - s_n \to 0$. By (b), we can choose cross-cuts C_n of $G(t_n)$ that separate 0 from $G(t_n) \setminus G(s_n)$ such that sph dm $C_n \to 0$. Let the cross-cuts Q_n of D be defined by $C_n = f(Q_n, t_n)$. Lemma B shows that diam $Q_n \to 0$. Since Q_n separates 0 from $A(s_n, t_n)$, it follows that

$$\text{diam } A(s_n, t_n) \to 0 . \tag{2.11}$$

Therefore Lemma C implies that, uniformly in D,

$$\phi(z, s_n, t_n) \to z . \tag{2.12}$$

ii) Let first $\alpha < t \leq \beta$. It follows from (2.10) and (2.11) that as $s \uparrow t$, the set $A(s, t)$ converges decreasingly to a point. Let $\zeta(t)$ be this limit point. Then $|\zeta(t)| = 1$, and $\zeta(t) \in \partial A(s, t)$ for $s < t$.

Let $t_n \to t$, and $s < t$, $s < t_n$. It follows from (2.9) and (2.12) that $A(s, t_n)$ converges to $A(s, t)$. Because $\zeta(t_n) \in \partial A(s, t_n)$, this implies that all limit points of $\zeta(t_n)$ as $n \to \infty$ lie in the closure of $A(s, t)$. This is true for every $s < t$. Hence $\zeta(t_n) \to \zeta(t)$. Consequently, $\zeta(t)$ is continuous for $\alpha < t \leq \beta$.

Let now $\alpha \leq s < t < \beta$ and $t \leq \tau \leq \beta$. We deduce from (2.9) and (2.12) that $A(s, \tau)$ converges to $A(s, t)$ as $\tau \downarrow t$. Since $A(t, \tau) \subset A(s, \tau)$ and $\zeta(t) \in \partial A(s, t)$ for every $s < t$, it follows that $A(t, \tau)$ converges to $\zeta(t)$ as $\tau \downarrow t$.

Finally, let $\alpha \leq \tau < \tau'$. As above, $A(\alpha, \tau)$ and $A(\alpha, \tau')$ differ by arbitrarily little if $\tau' - \alpha$ is sufficiently small. Therefore (2.11) implies that $A(\alpha, \tau)$ converges to a point as $\tau \downarrow \alpha$. We define $\zeta(\alpha)$ to be this limit point. Since $\zeta(\tau) \in \partial A(\alpha, \tau)$, it follows that $\zeta(\tau) \to \zeta(\alpha)$ as $\tau \downarrow \alpha$.

Thus we have obtained a function $\zeta(t)$, continuous on $[\alpha, \beta]$, with $|\zeta(t)| = 1$, and $A(t, t')$ (respectively, $A(t', t)$) tends to $\zeta(t)$ as $t' \to t$.

iii) Let $\alpha \leq s < t \leq \beta$. The function $\phi(z, s, t)$ maps D onto $D \setminus A(s, t)$. As in [7, p. 160] we define

$$h(z, s, t) = \frac{e^t + e^s}{e^t - e^s} \cdot \frac{z - \phi(z, s, t)}{z + \phi(z, s, t)} = 1 + \cdots .$$

Since $\phi(z, s, t) = e^{s-t} z + \cdots$, Lemma C shows that

$$h(z, s, t) \to \frac{\zeta(t) + z}{\zeta(t) - z} \quad (s \uparrow t),$$

respectively,

$$h(z, s, t) \to \frac{\zeta(s) + z}{\zeta(s) - z} \quad (t \downarrow s).$$

It follows [7, p. 163] that (1.6) is satisfied.

2.4. *Proof of the Corollary.* Let $\beta < \infty$, let $\alpha \leq s \leq t \leq \beta$, and let $w \in G(t) \cap \partial G(s)$. Because $w \in \partial G(s)$, we see that $w \in B(\tau)$ for some $\tau \geq s$. Because $w \in G(t)$ and $G(t) \cap B(t)$ is empty we conclude that $\tau < t$, hence $s \leq \tau < t$. It follows that

$$G(t) \cap \partial G(s) \subset \{b(\tau): s \leq \tau < t\}. \tag{2.13}$$

Corresponding to any $\varepsilon > 0$, we choose $\delta > 0$ such that $|b(\tau) - b(s)| < \varepsilon/2$ for $0 \leq t - s \leq \delta$. By (2.13), $\operatorname{diam}[G(t) \cap \partial G(s)] < \varepsilon/2$. Hence some cross-cut C of $G(t)$ with $\operatorname{diam} C < \varepsilon$ separates $G(t) \setminus G(s)$ from 0. Thus (b) holds, and therefore (a) is satisfied, by Theorem 1.

3. THE DISCONTINUOUS CASE

3.1. *Proof of Theorem* 2. Let $R = \{\omega_k: k = 1, 2, \cdots\}$ be a countable dense subset of the complement E of the image domain G. Since E is arcwise connected, some Jordan arc $J_k \subset E$ connects ω_k with ∞.

Let $E_1 = J_1$, and let G_1 denote the complement of E_1. Then G_1 is simply connected and $G_1 \supset G$. Let $w_1(\tau)$ $(0 \leq \tau < \infty)$ give a parametric representation of J_1 such that $w_1(\infty) = \infty$. The complements of the sets $\{w_1(\sigma): \sigma \geq \tau\}$ form a strictly increasing family of domains. We can say that the boundary J_1 has been continuously reduced to the point ∞. Let us consider the functions that map D onto these domains. After a suitable normalisation of the parameter, we may assume that the mapping functions have the form $f_1(z, t) = e^t z + \cdots$ $(\alpha_1 \leq t < \infty)$. By Loewner's slit-mapping theorem or by the corollary, we see that $f_1(z, t)$ satisfies (1.4), with

$$h_1(z, t) = (\zeta_1(t) + z)/(\zeta_1(t) - z) \qquad (\zeta_1(t) \text{ continuous in } [\alpha_1, \infty)).$$

Let the construction already be performed up to $k - 1$. Let $J_k^* = J_k$ if $J_k \cap E_{k-1}$ is empty. Otherwise, let J_k^* be the subarc of J_k from ω_k to the first intersection c_k with E_{k-1}. Possibly, $J_k^* = \{\omega_k\}$. Let $E_k = E_{k-1} \cup J_k^*$. Then the complement G_k of E_k is simply connected and $G_k \subset G_{k-1}$. As above, we reduce J_k^*

continuously either to ∞ or to c_k. We obtain a subordination chain $f_k(z, t) \in \mathscr{S}(\alpha_{k-1}, \alpha_k)$ that satisfies (1.4), with

$$h_k(z, t) = (\zeta_k(t) + z)/(\zeta_k(t) - z) \qquad (\zeta_k(t) \text{ continuous in } [\alpha_{k-1}, \alpha_k]).$$

We define

$$\zeta(t) = \zeta_k(t), \quad f(z, t) = f_k(z, t) \quad \text{for } \alpha_{k-1} \leq t < \alpha_k \qquad (k = 1, 2, \cdots).$$

Since R is a dense subset of E, it follows from Carathéodory's kernel theorem [1, p. 46] that $\alpha_k \to \alpha$. Therefore, if we put $f(z, \alpha) = f(z)$, we have a subordination chain $f(z, t) \in \mathscr{S}(\alpha, \infty)$. For $\alpha < t < \infty$ and $t \neq \alpha_k$ $(k = 1, 2, \cdots)$, $\zeta(t)$ is continuous and (1.7) is satisfied.

3.2. *Example.* Let S be the classical Cantor set in [0, 1], let $I_k = [2 \cdot 3^{-k}, 3 \cdot 3^{-k}]$ $(k = 1, 2, \cdots)$, and let H^+ and H^- denote the closed upper and lower half-planes. Through each point of S we draw a semi-circle in H^+ of center 1/2. Through each point of $S \cap I_k$ $(k = 1, 2, \cdots)$ we draw a semi-circle in H^- whose center is the midpoint $\frac{5}{2} \cdot 3^{-k}$ of I_k. The continuum E obtained this way was first described by Knaster (for a figure, see for instance [5, p. 143, Example 1]). It is indecomposable, that is, it cannot be decomposed into the union of two proper subcontinua.

For $n = 1, 2, \cdots$, let S_n be the finite set of all points in S that have the form $\nu/3^n$ $(\nu = 0, 1, 2, \cdots, 3^n)$, and let E_n consist of the union of the semi-circles in H^+ that have the center 1/2 and pass through a point of S_n, together with the semi-circles in H^- that pass through a point of $S_n \cap I_k$ and have the center $\frac{5}{2} \cdot 3^{-k}$ $(k = 1, \cdots, n)$. Then E_n is a Jordan arc from 0 to 3^{-n}, $E_n \subset E_{n+1}$, and E is the closure of the open Jordan arc

$$J = \bigcup_{n=1}^{\infty} E_n.$$

Let E^* and J^* be obtained from E and J, respectively, by the transformation $w^* = 1/w$. Then E^* is the closure of the open Jordan arc J^*. Hence the function f(z) that maps D onto the complement of E^* satisfies the assumptions of the following proposition (we can prove it by considering the complement of

$$J(\tau) = \{w(\sigma): \tau \leq \sigma < \infty\}$$

and proceeding as in the last part of the proof of Theorem 2):

Let $f(z) = e^{\alpha} z + \cdots$ map D one-to-one onto a domain whose complement is the closure of the open Jordan arc $J = \{w(\tau): 0 < \tau < \infty\}$ going to ∞. Then there exists an $f(z, t) \in \mathscr{S}(\alpha, \infty)$ such that $f(z, \alpha) = f(z)$ and (1.7) holds, where $\zeta(t)$ is continuous in the open interval (α, ∞).

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