COMMON PERIODIC POINTS OF COMMUTING FUNCTIONS

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INTRODUCTION

Let f and g be continuous transformations of $I = [0, 1]$ into itself that commute: $f(g(x)) = g(f(x))$ for all x in I. It is an unanswered question of considerable interest whether there always exists a common fixed point: $x = f(x) = g(x)$. An affirmative answer has been obtained under certain severe restrictions on f and g (see, for example, [1]). The purpose of this paper is to prove that there always exists a point x such that $x = f(x) = g^n(x)$ for some n, under the additional, mild assumption that f has a continuous derivative.

PRELIMINARIES

For any function h and any x in I, let $h^0 x = x$ and $h^k x = h(h^{k-1} x)$ for $k \geq 1$.

We shall consider the semigroup of transformations $(k, x) \to g^k x$ $(k = 0, 1, \cdots)$, where g is a continuous transformation of I into itself. By Ox we denote $\{g^k x: k \geq 0\}$, the *orbit of* x, and by Cx the closure of Ox. We shall say x is *periodic* if $g^n x = x$ for some $n > 0$. By P we denote the set of periodic points.

A subset Y of I is called *invariant* if gY is contained in Y. A closed, invariant, nonempty subset is called *minimal* if it contains no proper subset that is also closed, invariant, and nonempty.

A point x in I is called *recurrent* if x is in Cgx, that is, if $g^k x$ comes arbitrarily close to x for arbitrarily large values of k.

We shall assume that f is a transformation of I into itself and that it has a continuous derivative f'. We also assume that f commutes with g: fg = gf. By F we denote $\{x: fx = x\}$, the set of fixed points of f.

We do *not* consider the transformation semigroup generated by f, and for this reason we have dispensed with the prefix g in terms such as g-invariant, g-minimal, g-periodic.

We now consider some elementary facts concerning minimal sets.

PROPOSITION 1. *Every closed, invariant, nonempty subset contains a minimal set.*

Proposition 1 is proved by applying Zorn's Lemma.

PROPOSITION 2. *If* Y *is minimal and* y *is in* Y, *then* Cy = Y.

This follows from the fact that Cy is a closed, invariant, nonempty subset of Y.

PROPOSITION 3. *If* Y *is minimal and* y *is in* Y, *then* y *is recurrent.*

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PROPOSITION 4. *If* Y *is minimal and is not the orbit of a periodic point, then* Y *is perfect.*

Since each y in Y is in Cgy but not in Ogy, no y in Y is isolated.

PROPOSITION 5. F *contains a minimal set.*

If x is in F, then gx = gfx = fgx is also in F. Thus F is a closed, invariant, nonempty set.

PROPOSITION 6. *If* Y *is a perfect subset of* F, *then* f' y = 1 *for all* y *in* Y.

Let $\{y_n\}$ be a sequence in Y - y such that $y_n \to y$. Then

$$f'y = \lim[(fy_n - fy)/(y_n - y)] = 1 .$$

A THEOREM ON MINIMAL SETS

The following theorem may be deduced from a result of Jewett [2]. We give here a proof that is much shorter and avoids ergodic theory.

THEOREM 1. *Every minimal set is contained in the closure of* P.

Proof. Let Y be a minimal set. If Y is the orbit of a periodic point, then the conclusion is obviously valid. We thus consider the case where Y is not a periodic orbit.

Let $\varepsilon > 0$ be given. Let b = inf Y. Since b is in Y and Y = Cgb, there exist integers N and M such that $b < g^{N+M}b < g^N b < b + \varepsilon$.

Since Y is minimal and is not a periodic orbit, $g^M b > b$. It follows from the continuity of g^M that there exists a point e in $(b, g^N b)$ such that $g^N e = e$. Since $b < e < b + \varepsilon$, we have established the existence of periodic points arbitrarily close to b.

Now, for each y in Y, it follows from Proposition 2 that $|g^K b - y| < \varepsilon/2$ for some integer K. Since g^K is continuous, there exists an $\varepsilon' > 0$ such that $|g^K x - g^K b| < \varepsilon/2$ if $|x - b| < \varepsilon'$. From what we have shown above, it follows that there exist a point z and an integer L such that $|b - z| < \varepsilon'$ and $g^L z = z$. Thus $g^K z$ is a periodic point at a distance less than ε from y.

COROLLARY. *A minimal set is nowhere dense.*

THE MAIN THEOREM

THEOREM 2. *Let* Y *be a minimal subset of* F. *Then* Y *is contained in the closure of* $(P \cap F)$.

Proof. If Y is a periodic orbit, we have finished. If Y is not a periodic orbit, it is perfect, and f' y = 1 for all y in Y, according to Propositions 4 and 6.

Let $\varepsilon > 0$ be given. We may assume that ε is so small that $|f'u - f'v| < 1/2$ if $|u - v| < \varepsilon$.

Choose y_1 and y_2 in Y and x such that $y_1 < x < y_2 < y_1 + \varepsilon$ and $g^n x = x$ for some n. If fx = x, we have finished. If $fx \neq x$, choose z_1 and z_2 in F so that $y_1 \leq z_1 < x < z_2 \leq y_2$ and (z_1, z_2) contains no point of F. Thus fw > w for all w in (z_1, z_2) or $fw < w$ for all w in (z_1, z_2). We may assume without loss of generality that $fw > w$. On the other hand,

$$fz_2 - fw = z_2 - fw = f'w' \cdot (z_2 - w)$$

for some w' in (w, z_2), and

$$f'w' > f'z_2 - \frac{1}{2} = \frac{1}{2},$$

so that $fw < z_2$.

Thus $\{f^k x\}$ is an increasing sequence in (z_1, z_2) with a limit ℓ. Since $f\ell = \lim f^{k+1} x = \ell$, ℓ is in F. In fact, $\ell = z_2$. On the other hand,

$$g^n f^k x = f^k g^n x = f^k x \qquad \text{for each } k,$$

so $g^n \ell = \ell$.

Thus we have found an element ℓ, such that $g^n \ell = \ell$ and $f\ell = \ell$, lying within ε of y_1 and y_2. Since y_1 or y_2 could be any element of Y, the proof is complete.

COROLLARY. *If* f *is a* C^1*-function that commutes with a continuous function* g, *then there exists a point* x *such that* $fx = x = g^n x$ *for some integer* n.

Proof. According to Proposition 5, F contains a minimal set Y, which by definition is not empty. Thus, according to Theorem 2, there is a point y in $P \cap F$. It follows from the definition of P and F that $fx = x = g^n x$ for some integer n.

REFERENCES

1. H. Cohen, *On fixed points of commuting functions*, Proc. Amer. Math. Soc. 15 (1964), 293-296.

2. R. I. Jewett, *Invariant measures and periodic points*, (to appear).

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