FIBRING WITHIN A COBORDISM CLASS

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1. INTRODUCTION

The following question was pointed out to us by T. E. Stewart. If a closed manifold M^n is fibred differentiably by a closed connected fibre that cobords modulo 2, over the circle as base space with structure group Z_2, then does M^n cobord modulo 2? We can exhibit a 4-manifold M^4 fibred over the circle by a closed connected 3-manifold, which must cobord, but where $[M^4]_2$ is an indecomposable cobordism class. This led us to the problem of characterizing those unoriented cobordism classes that admit a representative, differentiably fibred over S^1, with structure group Z_2. We consider the ring homomorphism $\chi: \mathfrak{N} \to Z_2$ that assigns to each closed manifold its Euler characteristic reduced modulo 2. This is an invariant of the modulo-2 cobordism class, since $\chi(V^n)$ (mod 2) equals

$$\langle w_n, \sigma_n \rangle \in Z_2,$$

where $w_n \in H^n(V^n; Z_2)$ is the nth Stiefel-Whitney class. If a manifold can be fibred over S^1, then its cobordism class lies in the kernel of χ, since the Euler characteristic modulo 2 of the total space is the product of the Euler characteristic of the base with that of the fibre. In (4.5) we show the converse; that is, a closed manifold M^n is cobordant to the total space of a differentiable fibration over S^1 with structure group Z_2 if and only if $\chi(M^n) \equiv 0$ (mod 2).

Two generalizations of this result are then made. The kernel of χ is an ideal in $\mathfrak{N}$. The correspondence

$$[M^n]_2 \to ([M^n]_2)^{2^k}$$

is a ring homomorphism in $\mathfrak{N}$. The image of the kernel of χ under this homomorphism is a subring $\mathfrak{N}(k) \subset \mathfrak{N}$. Any cobordism class in the ideal generated by $\mathfrak{N}(1)$ can be represented by a closed manifold fibred over S^2 with structure group U(1), while any cobordism class in the ideal generated by $\mathfrak{N}(2)$ can be represented as the total space of a differentiable fibration over S^4 with structure group Sp(1). We have no information about fibrations over S^8.

We use the notation $[B, X, F, \pi; G]$ for a fibre bundle with total space B, base X, fibre F, and structure group G. In Section 8 we show that if

$$[M^m, S^n, F^{m-n}, \pi; O(k)]$$

is a differentiable bundle and $n \neq 0, 1, 2, 4$, or 8, then $[M^m]_2 = 0$. This leads to the obvious conjecture that the result may hold if O(k) is replaced by any compact connected Lie group.

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Our present methods rely heavily on earlier results [2, Chapter IV, p. 59]. In particular, we established a device for analyzing the cobordism class of a closed manifold in terms of the normal bundle of the fixed point set of an involution on the manifold. This is the basic principle used in the present note.

An additional remark about the fibration $[M^4, S^1, F^3, \pi; Z_2]$ mentioned earlier is in order. The map $\pi: M^4 \to S^1$ defines a bordism class $[M^4, \pi]_2 \in \mathfrak{N}_4(S^1)$ (see [2]). But $\pi: M^4 \to S^1$ turns out to be bordant to the constant map of M^4 into S^1. The observation we want to make is that $[M^4]_2$ is indecomposable, that the fibre map has no critical points, but that every point of M^4 under the constant map is critical. This suggests that the critical (or singular) points of a smooth map do not depend to any significant extent on the bordism class of the map.

2. COMPUTATIONS

This section develops some basic lemmas that arise from the computation of the Stiefel-Whitney numbers of a few specific manifolds. Let $\xi \to V^m$ be a real, differentiable k-plane bundle over a closed manifold. We denote by $B(\xi) \to V^m$ the associated (k - 1)-sphere bundle and by $p: RP(\xi) \to V^m$ the associated bundle with fibre RP(k - 1) [2, p. 60]. We note that $B(\xi)$ is naturally a 2-fold covering of $RP(\xi)$, and we let $c \in H^1(RP(\xi); Z_2)$ be the characteristic class of this covering. Since the fibre in $RP(\xi) \to V^m$ is totally nonhomologous to zero,

$$p^*: H^*(V^m; Z_2) \to H^*(RP(\xi); Z_2)$$

is injective. If $1, v_1, \cdots, v_k$ denote the Whitney class of $\xi \to V^m$, then the cohomology ring of $H^*(RP(\xi); Z_2)$ is entirely determined by the relation

$$c^k = \sum_1^k p^*(v_j) c^{k-j}.$$

It is also possible to express the total Stiefel-Whitney class of the tangent bundle to $RP(\xi)$, which is a closed $(m + k - 1)$-manifold. Let $1, w_1, \cdots, w_m$ be the Stiefel-Whitney classes of V^m; then (see [2, (23.3)]) the total Stiefel-Whitney class of $RP(\xi)$ is

$$\left(\sum_0^m p^*(w_j)\right)\left(\sum_0^k (1 + c)^{k-j} p^*(v_j)\right).$$

We shall denote by $\xi \oplus nR \to V^m$ the sum of ξ with a trivial n-plane bundle.

(2.1) LEMMA. *If* $\xi \to V^m$ *is a differentiable* 2*-plane bundle, then* $[RP(\xi)]_2 = 0$ *in* $\mathfrak{N}_{m+1}$.

Proof. In this case $c^2 = cp^*(v_1) + p^*(v_2)$; thus

$$(1 + c)^2 + (1 + c)p^*(v_1) + p^*(v_2) = 1 + p^*(v_1).$$

Each Stiefel-Whitney class of $RP(\xi)$ is given by $W_i = p^*(w_i + w_{i-1} v_1)$. From dimensional considerations it follows immediately that every Stiefel-Whitney number of the $(m + 1)$-manifold $RP(\xi)$ is zero.

Next we focus our attention on the canonical twisted real line bundle $\xi \to RP(1)$.

(2.2) LEMMA. *For any* $n \geq 0$, $[RP(\xi \oplus nR)]_2 = 0$ *in* $\mathfrak{N}_{n+1}$.

Proof. Again we must show that every Stiefel-Whitney number vanishes. Let $d \in H^1(RP(1); Z_2)$ be the generator. The Stiefel-Whitney classes of the $(n+1)$-manifold $RP(\xi \oplus nR)$ are

$$W_j = \binom{n+1}{j} c^j + \binom{n}{j-1} c^{j-1} p^*(d),$$

and $c^{n+1} = c^n p^*(d)$. Being concerned only with $j > 0$, we write

$$W_j = \binom{n+1}{j} c^j + \frac{j}{n+1}\binom{n+1}{j} c^{j-1} p^*(d).$$

Choosing positive integers j_r such that $j_1 + j_2 + \cdots + j_k = n+1$, and noting that $d^2 = 0$, we have the relation

$$W_{j_1} \cdots W_{j_k} = \prod_1^k \binom{n+1}{j_r} c^{n+1} + \sum_1^k \frac{j_s}{n+1}\left(\prod_1^k \binom{n+1}{j_r} c^n p^*(d)\right)$$

$$= \left[\prod_1^k \binom{n+1}{j_r} + \sum_1^k \frac{j_s}{n+1} \prod_1^k \binom{n+1}{j_r}\right] c^n p^*(d).$$

Since $\sum_1^k j_s/(n+1) = 1$, the term in brackets is 0 modulo 2, so that every Stiefel-Whitney number vanishes, and $[RP(\xi \oplus nR]_2 = 0$.

Our next lemma will exhibit some odd-dimensional indecomposable cobordism classes. We recall that $[M^n]_2 \in \mathfrak{N}_n$ is indecomposable if and only if it cannot be expressed as a sum of products of lower-dimensional classes. To test for indecomposability we shall evaluate a certain Stiefel-Whitney number [3]. Briefly, we shall consider a closed manifold M^n whose total Stiefel-Whitney class can be expressed as a product $(1+t_1) \cdots (1+t_{n+k})$, where each t_i belongs to $H^1(M^n; Z_2)$. We shall evaluate the symmetric polynomial $\sum_1^{n+k} (t_i)^n$ on the fundamental cycle of M^n and show that it is 1 modulo 2.

We select an odd number n for which $n+1 = 2^p(2q+1)$ $(q > 0)$, and we let $\xi \to RP(2^p)$ be the canonical twisted real line bundle.

(2.3) THEOREM. *The unoriented cobordism class* $[RP(\xi \oplus (q2^{p+1} - 1)R)]_2 \in \mathfrak{N}_n$ *is indecomposable.*

Let $d \in H^1(RP(2^p); Z_2)$ be the generator. The total Stiefel-Whitney class of $RP(\xi \oplus (q2^{p+1} - 1)R)$ is

$$(1 + p^*(d))^{2^p+1} (1+c)^{q2^{p+1}-1} (1 + c + p^*(d)).$$

We are required to show that

$$(2^p + 1) p^*(d^n) + (q2^{p+1} - 1) c^n + (c + p^*(d))^n \neq 0.$$

For dimensional reasons, $d^n = 0$. Now $c^{q2^p} = c^{q2^{p+1} - 1} p^*(d)$, hence

$$(q2^{p+1} - 1) c^n = (q2^{p+1} - 1) c^{q2^{p+1} - 1} p^*(d^{2^p}) \neq 0 .$$

It remains then to show that $(c + p^*(d))^n = 0$. Again for dimensional reasons,

$$(c + p^*(d))^n = \sum_0^{2^p} \binom{n}{j} c^{n-j} p^*(d^j) .$$

We may write $n - j = (q2^{p+1} - 1) + (2^p - j)$, so that

$$c^{n-j} p^*(d^j) = c^{q2^{p+1} - 1} p^*(d^{2^p - j} d^j) = c^{q2^{p+1} - 1} p^*(d^{2^p}) .$$

Thus we must show that $\sum_0^{2^p} \binom{n}{i} \equiv 0 \pmod 2$. Now $n = q2^{p+1} - 1\,(2^p - 1)$, hence

$$\binom{n}{i} = 1 \pmod 2 \qquad (0 \leq i < 2^p),$$

and $\binom{n}{2^p} = 0 \pmod 2$; therefore $\sum_0^{2^p} \binom{n}{i} = 0 \pmod 2$, and (2.3) is proved.

We shall also need to consider a complex k-plane bundle $\xi \to V^m$ together with the associated bundle $p\colon CP(\xi) \to V^m$ with fibre $CP(k - 1)$. In this case the $(2k - 1)$-sphere bundle $B(\xi)$ is a principal $U(1)$-bundle over $CP(\xi)$. We let $c \in H^2(CP(\xi); Z_2)$ be the Chern class modulo 2 of this $U(1)$-bundle. According to [1, p. 517],

$$c^k = \sum_1^k p^*(v_j) c^{k-j} ,$$

where $1, v_1, \cdots, v_k$ are the Chern classes modulo 2 of $\xi \to V^m$. The total Stiefel-Whitney class of the closed $[2(k - 1) - m]$-manifold $CP(\xi)$ is

$$\left(\sum_0^m p^*(w_j) \right) \left(\sum_0^k p^*(v_j) (1 + c)^{k-j} \right),$$

where $1, w_1, \cdots, w_m$ are the Stiefel-Whitney classes of V^m.

Finally, if $\xi \to V^m$ is a quaternionic k-plane bundle with structure group $Sp(k)$, then there exists a $QP(\xi) \to V^m$ with fibre $QP(k - 1)$. Analogous results hold for $H^*(QP(\xi); Z_2)$ and for the Stiefel-Whitney classes of $QP(\xi)$.

3. THE MODULE $I_*(Z_2)$

In the following brief discussion of the unoriented bordism group of all involutions on closed n-manifolds [2, p. 75], we shall identify involutions that are equivariantly diffeomorphic. An involution on a closed manifold (T, M^n) is said to bord if and only if there exists an involution on a compact manifold (τ, B^{n+1}) for which

$\tau \mid \partial B^{n+1}$ is equivariantly diffeomorphic to (T, M^n). From two involutions on closed manifolds (T_1, M_1^n) and (T_2, M_2^n) we may form a disjoint union

$$(T, M_1^n \cup M_2^n) \qquad (M_1^n \cap M_2^n = \emptyset)$$

in which each M_i^n is invariant $(i = 1, 2)$ and $T \mid M_i^n = (T_i, M_i^n)$. We shall say that (T_1, M_1^n) is bordant to (T_2, M_2^n) if and only if their disjoint union bords. This defines an equivalence relation among involutions on closed manifolds. We denote the equivalence class of (T, M^n) by $\{T, M^n\}$, and the collection of all such equivalence classes by $I_n(Z_2)$. By means of disjoint union we impose on $I_n(Z_2)$ an abelian group structure in which every element has order 2. It was shown in [2, (28.1)] that $I_n(Z_2)$ is a rather large, but finite group. Perhaps more to the point is the graded $\mathfrak{N}$-module structure on $I_*(Z_2) = \sum_0^\infty I_n(Z_2)$. For any involution (T, M^n) and any closed manifold V^m we form $(T', M^n \times V^m)$ by $T'(x, y) = (T(x), y)$ and introduce the $\mathfrak{N}$-module structure by

$$\{T, M^n\} \cdot [V^m]_2 = \{T', M^n \times V^m\}.$$

It turns out that $I_*(Z_2)$ is a free $\mathfrak{N}$-module.

We are especially interested in an $\mathfrak{N}$-module homomorphism $K_m: I_*(Z_2) \to \mathfrak{N}$ of degree m. We choose a representative (T, M^n) of $\{T, M^n\}$ and form the product $S^m \times M^n$, on which there exists the fixed-point-free involution $T_1(x, y) = (A(x), T(y))$, where (A, S^m) is the antipodal map. Let

$$K_m(\{T, M^n\}) = [(S^m \times M^n)/T_1]_2 \in \mathfrak{N}_{n+m}.$$

The reader may verify that this is a well-defined $\mathfrak{N}$-module homomorphism. The image of K_m is an ideal of $\mathfrak{N}$. We note that $(S^m \times M^n)/T_1$ is the bundle over the base space $RP(m)$ with fibre M^n and structure group Z_2 associated with the principal bundle $S^m \to RP(m)$. In other words, the image of K_m is the ideal of cobordism classes that contain a representative fibred over $RP(m)$ with group Z_2. We shall denote the manifold $(S^m \times M^n)/T_1$ by $P(m, n)$.

We denote by F the set of fixed points of an involution (T, M^n) on a closed manifold M^n. This fixed point set is the finite, disjoint union of closed connected submanifolds. By $\eta \to F$ we shall denote the normal bundle, with the understanding that the dimension of the fibre may vary from component to component. In any case it will be recognized that $RP(\eta \oplus R)$ is the finite, disjoint union of closed, connected n-manifolds. We shall use the fact (see [2, (24.2)]) that $[M^n]_2 = [RP(\eta \oplus R)]_2$, and we point out that if $F^1 \subset F$ is the union of the components of codimension 1 in M^n, then η restricted to F^1 is a line bundle $\eta_1 \to F^1$. But then $\eta_1 \oplus R$ is a 2-plane bundle, so that by (2.1) $[RP(\eta_1 \oplus R)]_2 = 0$.

(3.1) REMARK. *If* $[M^n]_2$ *is computed from the normal bundle to the fixed point set of an involution on* M^n, *then* F^1 *makes no contribution to* $[RP(\eta \oplus R)]_2 = [M^n]_2$.

This means we shall ignore F^1 in applications of [2, (24.2)].

4. THE IMAGE OF K_1

In this section we show that the image of K_1 coincides with the kernel of χ. With each involution (T, M^n) we shall first associate, by inductive definition, a sequence of involutions on closed manifolds $(\tau_k, V(n, k))$. Let

$$(\tau_0, V(n, 0)) = (T, M^n),$$

and suppose $(\tau_k, V(n, k))$ has been defined. On $S^1 \times V(n, k)$, define

$$T_1(z, x) = (-z, \tau_k(x)),$$

and let $V(n, k+1) = (S^1 \times V(n, k))/T_1$. We introduce T_2 on $S^1 \times V(n, k)$ by $T_2(z, x) = (\bar{z}, x)$. Since T_1 and T_2 commute, an involution $(\tau_{k+1}, V(n, k+1))$ is induced by T_2. From our definition we see that $V(n, k+1)$ is fibred by $V(n, k)$ over $RP(1)$.

Let $((z, x)) \in V(n, k+1)$ denote the equivalence class under T_1 of $(z, x) \in S^1 \times V(n, k)$. If

$$\tau_{k+1}((z, x)) = ((\bar{z}, x)) = ((z, x)),$$

then either $(z, x) = (\bar{z}, x)$ or $(-z, \tau_k(x)) = (\bar{z}, x)$. Therefore, to analyze the fixed point set F_{k+1} of τ_{k+1} we note that

(i) the fixed point set of T_2 is $S_1 = (\{1\} \cup \{-1\} \times V(n, k))$,

(ii) the set of coincidences of T_2 and T_1 is $S_2 = (\{i\} \cup \{-i\} \times F_k)$.

Here $F_k \subset V(n, k)$ is the fixed point set of τ_k. The fixed point set of τ_{k+1} is the disjoint union of S_1/T_1 with S_2/T_1. We identify the manifold $V(n, k)$ with S_1/T_1 by $x \to ((1, x))$, and F_k with S_2/T_1 by $x \to ((i, x))$. To understand the normal bundle $\eta_{k+1} \to F_{k+1}$, note first that $\eta_{k+1} \mid S_1/T_1$ is a trivial line bundle. To understand $\eta_{k+1} \mid S_2/T_1$, we should embed $V(n, k)$ again by $x \to ((i, x))$. Now this embedding also has a trivial normal line bundle; but $S_2/T_1 = F_k \subset V(n, k)$, so that $\eta_{k+1} \mid S_2/T_1$ is $\eta_k \oplus R \to F_k$, where η_k is the normal bundle to F_k in $V(n, k)$.

Using induction on k, we see that in fact F_{k+1} is a disjoint union

$$F \cup \left(\bigcup_0^k (V(n, j)) \right).$$

Furthermore,

$$\eta_{k+1} \mid F = \eta \oplus (k \oplus 1)R \to F,$$

where $\eta \to F$ is the original normal bundle to F in M^n, while $\eta_{k+1} \mid V(n, j)$ is a trivial $(k - j + 1)$-plane bundle. We can now use [2, (24.2)].

(4.1) LEMMA. *If* $\eta \to F$ *is the normal bundle to the fixed point set of* (T, M^n), *then*

$$[V(n, k)]_2 = [RP(\eta \oplus (k+1)R)]_2 + \sum_0^{k-1} [RP(k - j)]_2 [V(n, j)]_2 .$$

The following is an immediate corollary.

(4.2) LEMMA. *The cobordism class* $[V(n, k)]_2$ *is indecomposable if and only if* $[RP(\eta \oplus (k+1)R)]_2$ *is indecomposable.*

As we already noted, each $[V(n, k)]_2$ lies in the image of K_1. This brings us to a specific application.

(4.3) THEOREM. *To each integer* m $(m > 2,\ m \neq 2^p - 1)$ *there corresponds a differentiable fibring of closed manifolds* $[X^m, RP(1), F^{m-1}, \pi; Z_2]$ *for which* $[X^m]_2$ *is an indecomposable cobordism class.*

We begin with $m = 2s$, $s > 1$. We take $n = 2$, $k = 2(s - 1)$, and consider $V(2, 2(s - 1))$ arising from the involution $(T, RP(2))$ given by

$$T([x_1, x_2, x_3]) = [-x_1, x_2, x_3] .$$

The fixed point set F is the disjoint union of a point p and a projective line $RP(1) \subset RP(2)$. Furthermore, the restriction of $\eta \to F$ to RP(1) is the canonical, twisted, real line-bundle $\xi \to RP(1)$. Thus

$$[RP(\eta \oplus (2s - 1)R)]_2 = [RP(2s)]_2 + [RP(\xi \oplus (2s - 1)R)]_2 .$$

But by (2.2), $[RP(\xi \oplus (2s - 1)R)]_2 = 0$, and since 2s is even,

$$[RP(\eta \oplus (2s - 1)R)]_2 = [RP(2s)]_2$$

is indecomposable. By (4.2), we can take $X^{2s} = V(2, 2(s - 1))$.

Suppose next that $m + 1 = 2^p(2q + 1)$, $q > 0$. Let $(T, RP(2^p + 1))$ be the involution

$$T([x_1, \cdots, x_{2^p+2}]) = [-x_1, x_2, \cdots, x_{2^p+2}] .$$

We take $k = q2^{p+1} - 2$ and show that $[V(2^p + 1, q2^{p+1} - 2)]_2$ is indecomposable. The fixed point set of $(T, RP(2^p + 1))$ is the disjoint union of a point with an $RP(2^p)$. Furthermore, $\eta \to F$ restricted to $RP(2^p)$ is the canonical line bundle $\xi \to RP(2^p)$. Therefore

$$[RP(\eta \oplus (q2^{p+1} - 1)R)]_2 = [RP(m)]_2 + [RP(\xi \oplus (q2^{p+1} - 1)R)]_2 .$$

But $[RP(m)]_2 = 0$, since m is odd, so that $[RP(\eta \oplus (q2^{p+1} - 1)R)]_2$ is indecomposable. Again in view of (4.2) let $X^m = V(2^p + 1, q2^{p+1} - 2)$.

(4.4) REMARK. *The manifolds* V(2, k) *arising from the involution* (T, RP(2)) *cobord, for odd* k.

Write $k = 2s + 1$; then

$$[V(2, 2s + 1)]_2 = [RP(2s + 3)]_2 + [RP(\xi \oplus (2s+2)R)]_2 + \sum_0^{2s} [RP(2s + 1 - j)]_2 [V(2, j)]_2 .$$

Now $[RP(2s + 3)]_2 = [RP(\xi \oplus (2s+2)R)]_2 = 0$. Also, $[RP(2s + 1 - j)]_2 = 0$ if j is even, hence

$$[V(2, 2s+1)]_2 = \sum_0^{s-1} [RP(2(s-j)]_2 [V(2, 2j+1)]_2 .$$

We can now use induction on s, since for $s = 0$, $[V(2, 1)]_2 \in \mathfrak{N}_3 = 0$. This last means that X^{2s} is fibred over the circle by a closed connected fibre that cobords modulo 2.

We may think of $\mathfrak{N}$ as the graded polynomial algebra over Z_2 generated by $[RP(2)]_2$ together with the classes $[X^m]_2$ of (4.3). Any odd-dimensional class lies in the ideal generated by the $[X^m]_2$, while $[V^{2s}]_2 = [Y^{2s}]_2 + r[RP(2)]^s$, where $[Y^{2s}]_2$ lies in the ideal spanned by the $[X^m]_2$. Since $\chi([Y^{2s}]_2) = 0$, we see that $r = \chi(V^{2s})$, and we conclude that the ideal spanned by the $[X^m]_2$, the image of K_1, and the kernel of $\chi: \mathfrak{N} \to Z_2$ all coincide.

(4.5) THEOREM. *The cobordism class of a closed manifold admits a representative fibred over the circle with structure group* Z_2 *if and only if* $\chi(V^n) = 0$ (mod 2).

5. THE IMAGE OF K_m

In this section we present some partial results concerning the homomorphisms $K_m: I_*(Z_2) \to \mathfrak{N}$, for $m > 1$. We recall that from (T, M^n) we constructed $P(m, n) = (S^m \times M^n)/T_1$.

(5.1) LEMMA. *If* $\eta \to F$ *is the normal bundle to the fixed point set of* (T, M^n), *then* $[P(m, n)]_2 = [RP(\eta \oplus (m+1)R)]_2$.

To see this we examine the fixed point set of an additional involution $(\tau, P(m, n))$. First let (r, S^m) be the reflection involution

$$r(x_1, x_2, \cdots, x_{m+1}) = (-x_1, x_2, \cdots, x_{m+1}),$$

and define T_2 on $S^m \times M^n$ by $T_2(x, y) = (r(x), y)$. Since $T_1(x, y) = (A(x), T(y))$, we see that T_1 and T_2 commute, so that T_2 induces an involution $(\tau, P(m, n))$. To obtain the fixed point set of τ, note that

(i) the fixed point set of T_2 is $S^{m-1} \times M^n$,

(ii) the set of coincidences of T_1 and T_2 is $S^0 \times F$.

Again it is clear that the fixed point set of τ is the disjoint union of

$$(S^{m-1} \times M^n)/T_1 = P(m-1, n)$$

with $(S^0 \times F)/T_1 = F$. According to (3.1), we can ignore $P(m-1, n)$ in using ([2, (24.2)]). The normal bundle to $F \subset P(m, n)$ is $\eta \oplus mR \to F$, hence

$$[P(m, n)]_2 = [RP(\eta \oplus (m+1)R)]_2.$$

We now combine (5.1) with the involutions $(\tau_k, V(n, k))$ of Section 4. The fixed point set $F_k \subset V(n, k)$ is a disjoint union $F \cup \left(\bigcup_0^{k-1} V(n, j)\right)$, while

$$\eta_k \mid F = \eta \oplus kR$$

and $\eta_k \mid V(n, j)$ is a trivial $(k - j)$-plane bundle. We wish to apply (5.1) to $(S^m \times V(n, k))/T_1$.

(5.2) LEMMA. *If* $\eta \to F$ *is the normal bundle to the fixed point set of* (T, M^n), *then*

$$\begin{aligned}[P(m, n+k)]_2 &= [(S^m \times V(n, k))/T_1]_2 \\ &= [RP(\eta \oplus (m+k+1)R]_2 + \sum_0^{k-1} [RP(m+k-j)]_2 [V(n, j)]_2 .\end{aligned}$$

We note explicitly that $[P(m, n+k)]_2$ is indecomposable if and only if $[RP(\eta \oplus (m+k+1)R)]_2$ is indecomposable.

(5.3) THEOREM. *If* $2s \geq m$ *and* $2s \neq m+1$, *then there exists a differentiable fibring of closed manifolds* $[X^{2s}(m), RP(m), F^{2s-m}, \pi; Z_2]$ *for which* $[X^{2s}(m)]_2$ *is indecomposable.*

If $m = 2s$, we use $RP(2s)$ for the manifold. If $2s > m+1$, we take $n = 2$ and $k = 2(s-1) - m \geq 0$, and use the involution $(T, RP(2))$, recalling that

$$[RP(\eta \oplus (m+k+1)R]_2 = [RP(2s)]_2 .$$

We take $X^{2s}(m) = P(m, 2 + 2(s-1) - m)$ for the required manifolds.

(5.4) THEOREM. *If* $2s = 2^p(2q+1)$, $q > 0$, *and* $m \leq q2^{p+1} - 2$, *then there exists a differentiable fibring* $[X^{2s-1}(m), RP(m), F^{2s-1-m}, \pi; Z_2]$ *for which* $[X^{2s-1}(m)]_2$ *is indecomposable.*

Obviously, we use the involution $(T, RP(2^p + 1))$, so that

$$n = 2^p + 1 \quad \text{and} \quad k = 2s - 1 - m - n = q2^{p+1} - 2 - m .$$

Since $m \leq q2^{p+1} - 2$, it follows that $k \geq 0$, and we may apply (5.2).

This provides some indecomposable cobordism classes in the image of K_m. We do not know, however, whether (5.4) holds for some values of $m > q2^{p+1} - 2$. The first question is whether there exists a differentiable fibring $[X^5(3), RP(3), F^2, \pi; Z_2]$ for which $[X^5(3)]_2$ is indecomposable.

In concluding this section we note that if $m = 2$, then $2 \leq q2^{p+1} - 2$ for all $p \geq 1$ and $q > 0$; thus (5.4) applies to all odd numbers $2s - 1 \neq 2^p - 1$. Together with (5.3) this means that $\mathfrak{N}$ can be thought of as the polynomial algebra over Z_2 generated by $[X^s(2)]_2$ $(s \geq 2,\ s \neq 2^p - 1)$.

(5.5) COROLLARY. *The image of* $K_2 \colon I_*(Z_2) \to \mathfrak{N}$ *is the ideal* $\sum_2^\infty \mathfrak{N}_s$.

6. THE COMPLEX ANALOGUE

In the proof of (4.3) we used only some very special manifolds $V(n, k)$, namely, those that arise from an involution $(T, RP(n))$ given by

$$[x_1, x_2, \cdots, x_{n+1}] \to [-x_1, x_2, \cdots, x_{n+1}] .$$

In this section we introduce the complex analogues of these manifolds. Let T^k denote the k-dimensional toral group, and $\Sigma^k(C)$ the k-fold cartesian product of S^3 with itself. We shall express S^{2n+1} in complex coordinates $\lambda_1, \cdots, \lambda_{n+1}$ for which $\sum_1^{n+1} \lambda_i \bar{\lambda}_i = 1$. On $\Sigma^k(C) \times S^{2n+1}$ we define a principal action of T^{k+1} by setting

$$t((z_1, w_1), \cdots, (z_k, w_k), (\lambda_1, \cdots, \lambda_{n+1}))$$

$$= ((z_1 t_1^{-1}, w_1 t_1^{-1}), \cdots, (z_j t_j^{-1}, t_{j-1} w_j t_j^{-1}), \cdots, (z_k t_k^{-1}, t_{k-1} w_k t_k^{-1}),$$

$$(t_k \lambda_1 t_{k+1}^{-1}, \lambda_2 t_{k+1}^{-1}, \cdots, \lambda_{n+1} t_{k+1}^{-1})),$$

where $t = (t_1, \cdots, t_{k+1})$. We denote the quotient manifold by

$$(\Sigma^k(C) \times S^{2n+1})/T^{k+1} = \Gamma(n, k),$$

and a point in $\Gamma(n, k)$ by

$$[(z_1, w_1), \cdots, (z_k, w_k), (\lambda_1, \cdots, \lambda_{n+1})].$$

The mapping $\Gamma(n, k) \to CP(1)$ given by

$$[(z_1, w_1), \cdots, (z_k, w_k), (\lambda_1, \cdots, \lambda_{n+1})] \to [z_1, w_1]$$

is a fibre map with fibre $\Gamma(n, k-1)$ and structure group $U(1)$, as we can see by letting $U(1)$ act on $\Gamma(n, k-1)$ by

$$t_1([(z_2, w_2), \cdots, (z_k, w_k), (\lambda_1, \cdots, \lambda_{n+1})]$$

$$= [(z_2, t_1 w_2), \cdots, (z_k, w_k), (\lambda_1, \cdots, \lambda_{n+1})].$$

There is also the Hopf fibring $(S^3, U(1))$. The bundle with fibre $\Gamma(n, k-1)$ associated with the Hopf bundle has as total space $\Gamma(n, k)$.

Along the same lines we observe that the $\Gamma(1, k-1)$ play a special role. The map $\Gamma(n, k) \to \Gamma(1, k-1)$ given by

$$[(z_1, w_1), \cdots, (z_k, w_k), (\lambda_1, \cdots, \lambda_{n+1})] \to [(z_1, w_1), \cdots, (z_k, w_k)]$$

is a fibre map with fibre $CP(n)$ and structure group $U(1)$ acting on the fibre by

$$t_k([\lambda_1, \cdots, \lambda_{n+1}]) = [t_k \lambda_1, \lambda_2, \cdots, \lambda_{n+1}].$$

This means that there exists a complex line bundle $\xi \to \Gamma(1, k-1)$ such that the total space of $CP(\xi \oplus nC)$ is $\Gamma(n, k)$. This is the line bundle associated to the induced action of $U(1) = T^k/T^{k-1}$ on $(\Sigma^{k-1}(C) \times S^3)/T^{k-1}$, which is a principal $U(1)$-bundle over $\Gamma(1, k-1)$.

We could replace the toral group by $(Z_2)^{k+1}$. Let $\Sigma^k(R)$ be the k-fold product of S^1, and replace S^{2n+1} by S^n. If we use the analogous definition for the action of $(Z_2)^{k+1}$, we find that $(\Sigma^k(R) \times S^n)/(Z_2)^{k+1}$ is the manifold $V(n, k)$ arising from the involution $[x_1, x_2, \cdots, x_{n+1}] \to [-x_1, x_2, \cdots, x_{n+1}]$ on $RP(n)$. The comment is an immediate consequence of our definitions in Section 4. This is not an accident, for now we show that $[\Gamma(n, k)]_2 = ([V(n, k)]_2)^2$.

To do this we note that by our original definition $\Gamma(n, k)$ is a closed complex analytic $(n+k)$-manifold. We shall exhibit a conjugation involution on $\Gamma(n, k)$ (see [2, p. 63]) whose real fold (that is, set of fixed points) is $V(n, k)$. To do this we first define σ on $\Sigma^k(C) \times S^{2n+1}$ by

$$\sigma((z_1, w_1), \cdots, (z_k, w_k), (\lambda_1, \cdots, \lambda_{n+1}) = ((\bar{z}_1, \bar{w}_1), \cdots, (\bar{z}_k, \bar{w}_k), (\bar{\lambda}_1, \cdots, \bar{\lambda}_{n+1})).$$

If $t \to \bar{t}$ is the automorphism of T^{k+1} given by $(t_1, \cdots, t_{k+1}) \to (\bar{t}_1, \cdots, \bar{t}_{k+1})$, then $\sigma t = \bar{t}\sigma$. We also note that $\bar{t} = t^{-1}$; thus the subgroup of fixed elements is $(Z_2)^{k+1}$. The fixed point set of σ is $\Sigma^k(R) \times S^n = F$. If $tF \cap F \neq \emptyset$, then, since $\sigma t = \bar{t}\sigma$, we see that $t \in (Z_2)^{k+1}$ and $tF = F$. We may therefore identify the image of F under the quotient map

$$\Sigma^k(C) \times S^{2n+1} \to \Gamma(n, k) \quad \text{with} \quad (\Sigma^k(R) \times S^n)/(Z_2)^{k+1} = V(n, k).$$

We define the conjugation involution $(\sigma^*, \Gamma(n, k))$ by

$$[(z_1, w_1), \cdots, (z_k, w_k), (\lambda_1, \cdots, \lambda_{n+1})] \to [(\bar{z}_1, \bar{w}_1), \cdots, (\bar{z}_k, \bar{w}_k), (\bar{\lambda}_1, \cdots, \bar{\lambda}_{n+1})].$$

We must show that the fixed point set of σ^* is $V(n, k) \subset \Gamma(n, k)$; that is, if $p \in \Sigma^k(C) \times S^{2n+1}$ is a point for which $\sigma(p) = tp$ for some $t \in T^{k+1}$, then there must exist some $\tau \in T^{k+1}$ for which $\tau p \in \Sigma^k(R) \times S^n$. Since t has a square root, we can choose τ so that $\tau = \tau^{-1} t = \bar{\tau} t$; then $\sigma(\tau p) = \bar{\tau}\sigma(p) = \bar{\tau} tp = \tau p$, so that τp is fixed under σ. We may apply ([2, (24.4)]) to $(\sigma^*, \Gamma(n, k))$ and conclude that $[\Gamma(n, k)]_2 = ([V(n, k)]_2)^2$.

(6.1) THEOREM. *If* $m > 2$ *and* $m \neq 2^p - 1$, *then there exists a differentiable fibring of closed manifolds* $[Y^{2m}, CP(1), F^{2(m-1)}, \pi; U(1)]$ *for which* $[Y^{2m}]_2 = ([X^m]_2)^2$.

The manifolds X^m of (4.3) were certain ones of the $V(n, k)$. We use the corresponding manifold $\Gamma(n, k)$ for Y^{2m}. An immediate consequence:

(6.2) COROLLARY. *A cobordism class of the ideal in* $\mathfrak{N}$ *generated by squares of elements in the image of* K_1 *admits a representative fibred over the* 2*-sphere with structure group* $U(1)$.

The correspondence $[V^n]_2 \to ([V^n]_2)^2$ is a ring homomorphism. Therefore the ideal in question is generated by the squares $([X^m]_2)^2$; but for these classes we just proved the result in (6.1).

7. QUATERNION ANALOGUE

We can carry the constructions of Section 6 through one more stage. Let G^{k+1} denote the $(k+1)$-fold direct product of $Sp(1)$, and $\Sigma^k(Q)$ the k-fold product of S^7 with itself. Express S^{4n+3} in terms of quaternions $(\rho_1, \cdots, \rho_{n+1})$, where $\sum \rho_i \bar{\rho}_i = 1$. Now let G^{k+1} act on $\Sigma^k(Q) \times S^{4n+3}$ by

$$\begin{aligned} &t((q_1, p_1), \cdots, (q_k, p_k), (\rho_1, \cdots, \rho_{n+1})) \\ &\quad = ((q_1 t_1^{-1}, p_1 t_1^{-1}), \cdots, (q_j t_j^{-1}, t_{j-1} p_j t_j^{-1}), \cdots, (q_k t_k^{-1}, t_{k-1} p_k t_k^{-1})) \\ &\quad\quad (t_k \rho_1 t_{k+1}^{-1}, \rho_2 t_{k+1}^{-1}, \cdots, \rho_{n+1} t_{k+1}^{-1}). \end{aligned}$$

We denote the quotient manifold of this principal action by $\theta(n, k)$. Again, let $[(q_1, p_1), \cdots, (q_k, p_k), (\rho_1, \cdots, \rho_{n+1})]$ denote a point in $\theta(n, k)$. The map $\theta(n, k) \to QP(1)$ given by

$$[(p_1, q_1), \cdots, (p_k, q_k), (\rho_1, \cdots, \rho_{n+1})] \to [p_1, q_1]$$

is a fibre map with fibre $\theta(n, k-1)$ and structure group Sp(1). Also, the map $\theta(n, k) \to \theta(1, k-1)$ given by

$$[(q_1, p_1), \cdots, (q_k, p_k), (\rho_1, \cdots, \rho_{n+1})] \to [(q_1, p_1), \cdots, (q_k, p_k)]$$

is a QP(n)-bundle with Sp(1) acting on the fibre by

$$t_k([\rho_1, \cdots, \rho_{n+1}]) = ([t_k\rho_1, \rho_2, \cdots, \rho_{n+1}]).$$

There exists a quaternion line-bundle $\xi \to \theta(1, k-1)$ for which

$$QP(\xi \oplus nQ) = \theta(n, k).$$

In particular, $\theta(1, k) = QP(\xi \oplus Q)$.

There also exists a suitable involution. We let $\alpha: Q \to Q$ be the inner automorphism of period 2 in the quaternions that is given by $q \to iqi^{-1}$. The fixed subfield of α is the field of complex numbers. We let

$$\sigma((q_1, p_1), \cdots, (q_k, p_k), (\rho_1, \cdots, \rho_{n+1}))$$
$$= ((\alpha(q_1), \alpha(p_1)), \cdots, (\alpha(q_k), \alpha(p_k)), (\alpha(\rho_1), \cdots, \alpha(\rho_{n+1}))),$$

and let $\alpha(t) = (\alpha(t_1), \cdots, \alpha(t_{k+1}))$ denote the automorphism on G^{k+1}. Then $\sigma t = \alpha(t)\sigma$, the set F of fixed points of σ is $\Sigma^k(C) \times S^{2n+1}$, the fixed subgroup of α is $T^{k+1} \subset G^{k+1}$, and $tF \cap F \neq \emptyset$ if and only if $t \in T^{k+1}$ and $tF = F$. This means that

$$(\Sigma^k(C) \times S^{2n+1})/T^{k+1} = \Gamma(n, k)$$

can be identified with the image of F under the quotient map

$$\Sigma^k(Q) \times S^{4n+3} \to \theta(n, k).$$

Now we must show that if $(\sigma^*, \theta(n, k))$ is the involution induced by σ, then $\Gamma(n, k) \subset \theta(n, k)$ is the fixed point set of σ^*; that is, if a point is fixed under σ^*, we must be able to express it in complex coordinates.

Suppose now that

$$p = ((q_1, p_1), \cdots, (q_k, p_k), (\rho_1, \cdots, \rho_{n+1}))$$

is a point in $\Sigma^k(Q) \times S^{4n+1}$ for which $\sigma(p) = tp$ for some $t \in G^{k+1}$. By way of induction, let us assume that (q_i, p_i) are both complex for $1 \leq i \leq j-1$. Then

$$t_1 = \cdots = t_{j-1} = 1 \quad \text{and} \quad q_j t_j^{-1} = \alpha(q_j),\ p_j t_j^{-1} = \alpha(p_j).$$

There exists a $\tau_j \in Sp(1)$ for which $(q_j\tau_j^{-1}, p_j\tau^{-1})$ are complex. We replace p by

$(1, 1, \cdots, 1, \tau_j, 1, \cdots, 1)p$, which is equivalent to p under G^{k+1}. We begin with $j = 1$, and by induction we show that there exists a point p', equivalent to p, whose first k coordinate pairs are complex pairs. The reader may easily show that p' can further be replaced by a p" in which all coordinates are complex. This means that $\Gamma(n, k) \subset \theta(n, k)$ is the fixed point set of $(\sigma^*, \theta(n, k))$.

Unfortunately, in this case we do not know of a proposition analogous to [2, (24.2)]. It certainly seems that there should be an appropriate replacement. Anyway, we shall prove that the relation $[\theta(n, k)]_2 = ([\Gamma(n, k)]_2)^2$ still holds. For this we must analyze the cohomology rings

$$H^*(\Gamma(n, k); Z_2) \quad \text{and} \quad H^*(\theta(n, k); Z_2).$$

(7.1) LEMMA. *There exist cohomology classes* $\alpha_1, \cdots, \alpha_k$, c *in* $H^2(\Gamma(n, k); Z_2)$ *that generate the ring* $H^*(\Gamma(n, k); Z_2)$ *subject only to the relations*

$$\alpha_1^2 = 0, \quad \alpha_j^2 = \alpha_j\alpha_{j-1} \ (2 \leq j \leq k), \quad c^{n+1} = c^n \alpha_k.$$

The total Stiefel-Whitney class of $\Gamma(n, k)$ *is*

$$(1 + \alpha_1) \cdots (1 + \alpha_{k-1})((1 + c)^{n+1} + (1 + c)^n \alpha_k).$$

We consider the CP(n)-bundle $p: \Gamma(n, k) = CP(\xi \oplus nC) \rightarrow \Gamma(1, k - 1)$. The fibre is totally nonhomologous to zero modulo 2. If $\alpha \in H^2(\Gamma(1, k - 1); Z_2)$ is the first Chern class modulo 2 of $\xi \rightarrow \Gamma(1, k - 1)$, then $H^*(\Gamma(n, k); Z_2)$ is completely determined by the relation $c^{n+1} = p^*(\alpha)c^n$, where $c \in H^2(\Gamma(n, k); Z_2)$ is the Chern class modulo 2 of the principal U(1)-bundle $B(\xi \oplus nC) \rightarrow CP(\xi \oplus nC)$. We consider $\Gamma(1, k) \rightarrow \Gamma(1, k - 1)$ with fibre CP(1). In this case, $c \in H^2(\Gamma(1, k); Z_2)$ is the first Chern class modulo 2 of $\xi \rightarrow \Gamma(1, k)$. We suppose classes $\alpha_1, \cdots, \alpha_k$ in $H^2(\Gamma(1, k - 1); Z_2)$ are chosen that generate the ring $H^*(\Gamma(1, k - 1); Z_2)$ subject only to

$$\alpha_1^2 = 0, \quad \alpha_j^2 = \alpha_j\alpha_{j-1} \ (2 \leq j \leq k).$$

Finally we assume α_k is the Chern class modulo 2 of $\xi \rightarrow \Gamma(1, k - 1)$. Let $\alpha_j = p^*(\alpha_j)$ $(1 \leq j \leq k)$, and set $c = \alpha_{k+1}$. This furnishes us with appropriate classes in $\Gamma(1, k - 1)$. For $p: \Gamma(n, k) \rightarrow \Gamma(1, k - 1)$, take $\alpha_j = p^*(\alpha_j)$ $(1 \leq j \leq k)$, together with c. We now have the first part of (7.1).

Next we must show that the Stiefel-Whitney class of $\Gamma(1, k)$ is

$$(1 + \alpha_1) \cdots (1 + \alpha_k).$$

For $\Gamma(1, k)$, we note that

$$(1 + c)^2 + (1 + c)p^*(\alpha_k) = 1 + p^*(\alpha_k);$$

hence, by induction over k, we see that $p^*((1 + \alpha_1) \cdots (1 + \alpha_k))$ is the total Stiefel-Whitney class of $\Gamma(1, k)$. Applying this to $\Gamma(n, k) \rightarrow \Gamma(1, k - 1)$, we obtain the second part of (7.1).

If we think of $\Gamma(n, k)$ as the base space of a principal T^{k+1}-bundle, then the $\alpha_1, \cdots, \alpha_k$, c are simply the characteristic cohomology classes of the bundle. If $\theta(n, k)$ is considered as the base space of a principal G^{k+1}-bundle, then there are (modulo 2)-characteristic classes $\beta_1, \cdots, \beta_k$, d in $H^4(\theta(n, k); Z_2)$ that satisfy the relations analogous to those of (7.1).

Since $\Gamma(n, k) \subset \theta(n, k)$, we consider

$$i^*: H^*(\theta(n, k); Z_2) \rightarrow H^*(\Gamma(n, k); Z_2).$$

The homomorphism can be computed immediately, since

$$i^*(\beta_j) = \alpha_j^2 \ (1 \leq j \leq k) \quad \text{and} \quad i^*(d) = c^2.$$

According to (7.1), the image of the total Stiefel-Whitney class of $\theta(n, k)$ under i^* is then the square of the total Stiefel-Whitney class of $\Gamma(n, k)$. Upon application of the Whitney sum theorem, we see that the normal bundle $\eta \rightarrow \Gamma(n, k)$ has the same Whitney classes as the tangent bundle $\tau \rightarrow \Gamma(n, k)$. This means that $\tau \rightarrow \Gamma(n, k)$ is bordant to $\eta \rightarrow \Gamma(n, k)$ in terms of bundles; that is,

$$[\tau \rightarrow \Gamma(n, k)]_2 = [\eta \rightarrow \Gamma(n, k)]_2$$

in $\mathfrak{N}_{2(n+k)}(BO(2(n+k)))$ [2, p. 68]. An immediate corollary is that

$$[\theta(n, k)]_2 = [RP(\tau \oplus R)]_2 = [RP(\eta \oplus R)]_2 = [\Gamma(n, k) \times \Gamma(n, k)]_2 .$$

(7.2) THEOREM. *For* $m > 2$ *and* $m \neq 2^p - 1$, *there exists a differentiable fibring of closed manifolds* $[W^{4m}, QP(1), F^{4(m-1)}; \pi; Sp(1)]$ *for which*

$$[W^{4m}]_2 = ([X^m]_2)^4 .$$

Since the correspondence $[V^n]_2 \rightarrow ([V^n]_2)^4$ is a ring homomorphism in $\mathfrak{N}$, we also have the following result.

(7.3) COROLLARY. *A cobordism class of the ideal in* $\mathfrak{N}$ *generated by fourth powers of elements in the image of* K_1 *admits a representative fibred over the* 4-*sphere with structure group* SP(1).

8. A TECHNICAL ADDITION

(8.1) THEOREM. *If* $[M^m, S^n, F^{m-n}, \pi; O(k)]$ *is a differentiable fibration of closed manifolds, then* $[M^m]_2 = 0$ *if* $n \neq 0, 1, 2, 4$, or 8.

The bordism class of the bundle is an element of $\mathfrak{N}_n(BO(k))$ [2]. The bordism class of this bundle is uniquely determined by the Whitney numbers of the characteristic map $f: S^n \rightarrow BO(k)$ [2, (17.2)]. However, there is only one Whitney number of f that might be nonzero, namely the nth Whitney class $v_n \in H^n(S^n; Z_2)$. Milnor [4] has shown, however, that $v_n = 0$ if $n \neq 0, 1, 2, 4$, or 8 [4]. Hence, except in these cases, all Whitney numbers of the characteristic map of the bundle must vanish. But then $[M^m, S^n, F^{m-n}, \pi; O(k)]$ is bordant as a bundle to the product bundle $[S^n \times F^{m-n}, S^n, F^{m-n}, \pi ; O(k)]$, and in particular, $[S^n \times F^{m-n}]_2 = [M^m]_2 = 0$.

While it is clear that this argument would apply to some structural groups other than the orthogonal group, it is not certain what the best result is. An optimistic conjecture is that if $[M^m, S^n, F^{m-n}, \pi; G]$ is a differentiable fibring of closed manifolds in which the structure group is a compact, connected Lie group, then $[M^m]_2 = 0$ if $n \neq 0, 1, 2, 4$, or 8.

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