ON FUNCTIONS WITH FOURIER TRANSFORMS IN L_p

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1. INTRODUCTION

Let G and $\hat{G}$ be two locally compact abelian groups, in Pontrjagin duality. In this paper we intend to study the spaces $A_p(G)$ consisting of all complex-valued functions $f \in L_1(G)$ whose Fourier transforms $\hat{f}$ belong to $L_p(\hat{G})$ $(p \geq 1)$. It is quite clear that all the $A_p(G)$ are dense ideals in $L_1(G)$ under convolution. Further, if $f \in A_p(G)$, then $\hat{f}$ is a bounded continuous function in $L_p(\hat{G})$ and therefore belongs to $L_r(\hat{G})$ for all $r > p$. Thus we see that the $A_p(G)$ form an ascending chain of dense ideals in $L_1(G)$. We shall show that when they are endowed with suitable norms, the $A_p(G)$ become Banach algebras (Section 3). Further, their behavior is quite similar to that of $L_1(G)$. Thus they all have $\hat{G}$ as the space of maximal ideals (Section 3), spectral synthesis holds or fails for them according as it holds or fails for $L_1(G)$ (Section 4), and, in the noncompact case, they all have the Fourier-Stieltjes transforms as multipliers (Section 5).

We also include some other results: In Section 2 we give a description of the dual space of $A_p(G)$ as a Banach space. Finally, in Section 6 we show that $A_2(G) = L_1(G) \cap L_2(G)$, so that the results in C. R. Warner's dissertation (announced in [2]) are special, though prototypical, cases of ours.

We use Rudin [1] as our chief reference. We thank I. D. Berg for many helpful discussions.

2. BANACH SPACE STRUCTURE OF $A_p(G)$

For each p $(1 \leq p < \infty)$, set $\|f\|^p = \|f\|_1 + \|\hat{f}\|_p$ $(f \in A_p(G))$, where

$$\|f\|_1 = \int_G |f(x)|\, dx, \qquad \|\hat{f}\|_p = \left(\int_{\hat{G}} |f(y)|^p\, dy \right)^{1/p},$$

and dx and dy denote integration with respect to Haar measures on the groups G and $\hat{G}$, respectively.

THEOREM 1. *For each* p $(1 \leq p < \infty)$, $\| \ \|^p$ *is a complete norm for the space* $A_p(G)$; *that is,* $A_p(G)$ *is a Banach space.*

Proof. It is easy to verify that $\| \ \|^p$ is a norm.

Moreover, let $\{f_n\} \subset A_p(G)$ be a Cauchy sequence. Clearly, $\{f_n\}$ and $\{\hat{f}_n\}$ are then Cauchy sequences in $L_1(G)$ and $L_p(\hat{G})$, respectively, and so there exist functions $f \in L_1(G)$ and $g \in L_p(\hat{G})$ such that $\|f_n - f\|_1 \to 0$ and $\|\hat{f}_n - g\|_p \to 0$. However, since the Fourier transform is norm-decreasing, $\|\hat{f}_n - \hat{f}\|_\infty \to 0$, where

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$\|\hat{f}\|_\infty$ = ess. sup $|\hat{f}(y)|$, and since some subsequence of $\{\hat{f}_n\}$ converges to g almost everywhere, we must conclude that $\hat{f} = g$.

Therefore $\|f_n - f\|^p \to 0$, where $f \in A_p(G)$; that is, $\| \ \|^p$ is complete.

Next, consider for each p $(1 \leq p < \infty)$ the mapping

$$\Phi_p: A_p(G) \to L_1(G) \times L_p(\hat{G})$$

defined by $\Phi_p(f) = (f, \hat{f})$ $(f \in A_p(G))$. This is clearly a linear isometry of $A_p(G)$ into the Banach space $L_1(G) \times L_p(\hat{G})$ with the norm $\|(f, g)\| = \|f\|_1 + \|g\|_p$. Thus we consider $A_p(G)$ as a closed subspace of $L_1(G) \times L_p(\hat{G})$ $(1 \leq p < \infty)$.

Since the dual space of $L_1(G) \times L_p(\hat{G})$ is isomorphic to $L_\infty(G) \times L_q(\hat{G})$, where $L_\infty(G)$ is the Banach space of essentially bounded measurable functions with the essential supremum norm, and where $1/p + 1/q = 1$, a simple application of the Hahn-Banach theorem shows that each bounded linear functional F on $A_p(G)$ must be of the form

$$F(f) = \int_G f(x)\,\phi(x)\,dx + \int_{\hat{G}} \hat{f}(y)\,\psi(y)\,dy \qquad (f \in A_p(G)),$$

for some pair $(\phi, \psi) \in L_\infty(G) \times L_q(\hat{G})$. However, the pair (ϕ, ψ) corresponding to a given functional may not be unique.

The situation is described more precisely by Theorem 2. In the statement of this theorem, $A_p^*(G)$ denotes the dual space of $A_p(G)$ $(1 \leq p < \infty)$, and K_p consists of the pairs $(\phi, \psi) \in L_\infty(G) \times L_q(\hat{G})$ for which there exists a net $(\tilde{a}_\alpha, \hat{a}_\alpha)$, belonging to

$$B_p = \{(\tilde{a}, \hat{a}) \mid a \in L_1(G),\ \hat{a} \in L_1(\hat{G}) \cap L_q(\hat{G}),\ \tilde{a}(x) = -a(-x)\},$$

such that

$$\lim_\alpha \int_G f(x)\,\tilde{a}_\alpha(x)\,dx = \int_G f(x)\,\phi(x)\,dx,$$
$$\lim_\alpha \int_{\hat{G}} \hat{f}(y)\,\hat{a}_\alpha(y)\,dy = \int_{\hat{G}} \hat{f}(y)\,\psi(y)\,dy. \qquad (f \in A_p(G))$$

In other words, K_p is the closure of B_p in the weak topology induced by $A_p(G)$.

THEOREM 2. *For each* p $(1 \leq p < \infty)$, *the dual space* $A_p^*(G)$ *of* $A_p(G)$ *is isomorphic to* $L_\infty(G) \times L_q(\hat{G})/K_p$ $(1/p + 1/q = 1)$.

Proof. From the remarks preceding the theorem, it is clear that $A_p^*(G)$ is isomorphic to $L_\infty(G) \times L_q(\hat{G})/I_p$ for some kernel I_p. To establish the theorem we must show that $I_p = K_p$.

Let $(\tilde{a}, \hat{a}) \in B_p$. Then, for each $f \in A_p(G)$,

$$\int_G f(x)\tilde{a}(x)\,dx + \int_{\hat{G}} \hat{f}(y)\hat{a}(y)\,dy$$

$$= \int_G f(x)\tilde{a}(x)\,dx + \int_{\hat{G}} \left(\int_G (-x,\ y) f(x)\,dx\right)\hat{a}(y)\,dy$$

$$= \int_G f(x)\tilde{a}(x)\,dx + \int_G f(x)\left(\int_{\hat{G}} (-x,\ y)\hat{a}(y)\,dy\right)dx$$

$$= \int_G f(x)(\tilde{a}(x) + a(-x))\,dx = 0.$$

The interchange of the order of integration is permissible, since $\hat{a} \in L_1(\hat{G}) \cap L_q(\hat{G})$. Thus $B_p \subset I_p$.

Moreover, any pair $(\phi, \psi) \in L_\infty(G) \times L_q(\hat{G})$ that is the limit of a net in B_p, in the sense described before the statement of the theorem, obviously belongs to I_p; that is, $K_p \subset I_p$. Hence we need only show that no other pair (ϕ, ψ) can occur. For this it is clearly sufficient to show that if (ϕ, ψ) belongs to I_p, then for each $f \in A_p(G)$ and each $\varepsilon > 0$ we can find $(\tilde{a}, \hat{a}) \in B_p$ such that

(1) $$\left|\int_G f(x)\phi(x)\,dx - \int_G f(x)\tilde{a}(x)\,dx\right| < \varepsilon,$$

(2) $$\left|\int_{\hat{G}} \hat{f}(y)\psi(y)\,dy - \int_{\hat{G}} \hat{f}(y)\hat{a}(y)\,dy\right| < \varepsilon.$$

Since $(\phi, \psi) \in I_p$, the inequality (2) implies (1), and so we have only to show that (2) holds.

With this in mind we first note that $C_p = \{\hat{a} \mid (\tilde{a}, \hat{a}) \in B_p\}$ is a self-adjoint, separating subalgebra under pointwise multiplication of $C_0(\hat{G})$, the space of continuous functions on $\hat{G}$ vanishing at infinity; it follows from the Stone-Weierstrass theorem that C_p is norm-dense in $C_0(\hat{G})$. Hence C_p is also norm-dense in $L_q(\hat{G})$ $(q \neq \infty)$.

Thus for each $f \in A_p(G)$ $(p > 1)$ and each $\varepsilon > 0$ there exists an $\hat{a} \in C_p$ such that

$$\left(\int_{\hat{G}} |\psi(y) - \hat{a}(y)|^q\,dy\right)^{1/q} < \varepsilon/\|\hat{f}\|_p$$

and hence

$$\left|\int_{\hat{G}} \hat{f}(y)\psi(y)\,dy - \int_{\hat{G}} \hat{f}(y)\hat{a}(y)\,dy\right|$$

$$\leq \left(\int_{\hat{G}} |\hat{f}(y)|^p\,dy\right)^{1/p}\left(\int_{\hat{G}} |\psi(y) - \hat{a}(y)|^q\,dy\right)^{1/q} < \varepsilon.$$

Similarly, we can prove inequality (2) in the case $p = 1$, $q = \infty$ by using the fact that the functions in $L_\infty(\hat{G})$ can be uniformly approximated on compact sets by functions in C_1.

3. BANACH ALGEBRA STRUCTURE OF $A_p(G)$

THEOREM 3. *For each* p $(1 \leq p < \infty)$, $A_p(G)$ *is a Banach algebra under convolution with the norm* $\| \, \|^p$.

Proof. In the preceding section we showed that $A_p(G)$ is a Banach space with the norm $\| \, \|^p$.

Moreover, for each pair f, g in $A_p(G)$, $f * g \in L_1(G)$ since $f, g \in L_1(G)$ and

$$\|\widehat{f*g}\|_p = \|\hat{f}\hat{g}\|_p \leq \|\hat{f}\|_\infty \|\hat{g}\|_p .$$

The right-hand side of the inequality is finite, since $\hat{g} \in L_p(\hat{G})$. Thus $f * g \in A_p(G)$, in other words; $A_p(G)$ is closed under convolution.

Finally, for each $f, g \in A_p(G)$,

$$\begin{aligned} \|f * g\|^p &= \|f * g\|_1 + \|\hat{f}\hat{g}\|_p \leq \|f\|_1 \|g\|_1 + \|\hat{f}\|_\infty \|\hat{g}\|_p \\ &\leq \|f\|_1 (\|g\|_1 + \|\hat{g}\|_p) \\ &\leq (\|f\|_1 + \|\hat{f}\|_p)(\|g\|_1 + \|\hat{g}\|_p) = \|f\|^p \|g\|^p . \end{aligned}$$

Therefore $A_p(G)$ is a Banach algebra.

Remark. In the case $p = 1$, it is easy to see that $A_1(G)$ is also a Banach algebra under pointwise multiplication.

The next theorem identifies the space of maximal ideals of $A_p(G)$ as the dual group $\hat{G}$.

THEOREM 4. *For each* p $(1 \leq p < \infty)$, *the space of maximal ideals of* $A_p(G)$ *can be identified with the dual group* $\hat{G}$.

Proof. Let $f \in A_p(G)$, $f \neq 0$. Then, for each positive integer n,

$$\begin{aligned} \|f^n\|^p = \|f^{n-1} * f\|^p &= \|f^{n-1} * f\|_1 + \|\hat{f}^{n-1} \hat{f}\|_p \\ &\leq \|f^{n-1}\|_1 \|f\|_1 + \|\hat{f}^{n-1}\|_\infty \|\hat{f}\|_p \\ &\leq \|f^{n-1}\|_1 (\|f\|_1 + \|\hat{f}\|_p) \leq \|f\|_1^{n-1} \|f\|^p . \end{aligned}$$

Hence, for each n,

$$(\|f^n\|^p)^{1/n} \leq \|f\|_1^{(n-1)/n} (\|f\|^p)^{1/n} .$$

Letting n tend to infinity, we see that

$$\|f\|_{Sp}^p \leq \|f\|_1 \qquad (f \neq 0), \tag{3}$$

where $\| \|_{Sp}^{p}$ is the spectral radius norm in the algebra $A_p(G)$. Clearly (3) holds also in the case $f = 0$.

Hence, if F is any multiplicative linear functional on $A_p(G)$, then

$$|F(f)| \leq \|f\|_{Sp}^{p} \leq \|f\|_1 \qquad (f \in A_p(G));$$

that is, F defines an L_1-bounded multiplicative linear functional on $A_p(G)$ considered as a subspace of $L_1(G)$. Thus F may be extended to a multiplicative linear functional on all of $L_1(G)$, and the extension is unique, since $A_p(G)$ is norm-dense in $L_1(G)$.

Therefore, since the maximal ideal space of $L_1(G)$ is $\hat{G}$, there corresponds, to each multiplicative linear functional F on $A_p(G)$, a unique continuous character $(\cdot, y)$ on G such that

$$F(f) = \int_G (-x, y) f(x)\, dx \qquad (f \in A_p(G)),$$

and conversely. It is easy to verify that the usual topology on the dual group $\hat{G}$ coincides with the Gelfand topology on $\hat{G}$ considered as the space of multiplicative linear functionals.

Thus the maximal ideal space of $A_p(G)$ may be identified with $\hat{G}$.

4. IDEAL THEORY IN $A_p(G)$

In this section we shall show that there exists a one-to-one correspondence between the closed ideals in $L_1(G)$ and the closed ideals in $A_p(G)$. This is accomplished by the following theorem.

THEOREM 5. *For each* p $(1 \leq p < \infty)$ *the following two statements hold:*

i) *If* I_1 *is a closed ideal in* $L_1(G)$, *then* $I = I_1 \cap A_p(G)$ *is a closed ideal in* $A_p(G)$.

ii) *If* I *is a closed ideal in* $A_p(G)$ *and* I_1 *is the closure of* I *in* $L_1(G)$, *then* I_1 *is a closed ideal in* $L_1(G)$ *and* $I = I_1 \cap A_p(G)$.

Proof. The proof of i) is immediate and will be omitted. Similarly, in ii) it is easy to verify that I_1 is a closed ideal in $L_1(G)$ and that $I \subset I_1 \cap A_p(G)$.

Let $f \in I_1 \cap A_p(G)$. We must show that for each $\varepsilon > 0$ we can find a function $h \in I$ such that $\|h - f\|^p < \varepsilon$. Since $f \in I_1 \cap A_p(G)$, there exists a sequence $\{f_n\} \subset I$ such that $\|f_n - f\|_1 \to 0$. Let $\{u_\alpha\} \subset A_p(G)$ be an approximate identity in $L_1(G)$ for which

$$\|u_\alpha\|_1 \leq 1, \qquad 0 \leq |\hat{u}_\alpha(y)| \leq 1,$$

and $\hat{u}_\alpha$ has compact support for each α. It is then clear that

$$\|f_n * u_\alpha - f * u_\alpha\|_1 \to 0$$

as $n \to \infty$, uniformly in α. In particular, for each δ $(0 < \delta < 1)$ there exists an n_0 such that

(4) $\|f_{n_0} * u_\alpha - f\|_1 \leq \|f_{n_0} * u_\alpha - f * u_\alpha\|_1 + \|f * u_\alpha - f\|_1 < \frac{1}{4}\delta + \|f * u_\alpha - f\|_1$

for all α.

Next choose a compact set $K \subset \hat{G}$ such that

$$\int_{\sim K} |\hat{f}_{n_0}(y)|^p \, dy < \frac{\delta^p}{2^{2p+1}}, \qquad \text{and} \qquad \int_{\sim K} |f(y)|^p \, dy < \frac{\delta^p}{2^{2p+1}}.$$

This is possible, since f_{n_0}, $f \in A_p(G)$. Then, for all α,

$$\int_{\hat{G}} |\hat{f}_{n_0}(y)\hat{u}_\alpha(y) - \hat{f}(y)|^p \, dy$$

(5)
$$\leq \int_K |\hat{f}_{n_0}(y)\hat{u}_\alpha(y) - \hat{f}(y)|^p \, dy + \int_{\sim K} |\hat{f}_{n_0}(y)\hat{u}_\alpha(y) - \hat{f}(y)|^p \, dy$$
$$\leq \int_K |\hat{f}_{n_0}(y)\hat{u}_\alpha(y) - \hat{f}(y)|^p \, dy + 2^p \int_{\sim K} (|\hat{f}_{n_0}(y)|^p + |\hat{f}(y)|^p) \, dy$$
$$\leq \int_K |\hat{f}_{n_0}(y)\hat{u}_\alpha(y) - \hat{f}(y)|^p \, dy + \delta^p/2^p.$$

However, since $\|f * u_\alpha - f\|_1 \to 0$ over α, it is clear from (4) that we can choose an α_0 such that both

(6) $$\|\hat{f}_{n_0}\hat{u}_\alpha - \hat{f}\|_\infty^p < \delta^p/2^p$$

and

(7) $$\|f_{n_0} * u_\alpha - f\|_1 < \delta/2$$

hold for $\alpha \succ \alpha_0$.

Then, combining the inequalities (5) to (7), we see that

$$\|f_{n_0} * u_\alpha - f\|^p = \|f_{n_0} * u_\alpha - f\|_1 + \|\hat{f}_{n_0}\hat{u}_\alpha - \hat{f}\|_p$$
$$< \frac{1}{2}\delta + \frac{1}{2}\delta\,[m(K) + 1]^{1/p} \qquad (\alpha \succ \alpha_0),$$

where m denotes the Haar measure on $\hat{G}$.

Let $\delta < 2\varepsilon/\{1 + [m(K) + 1]^{1/p}\}$. Then $\|f_{n_0} * u_\alpha - f\|^p < \varepsilon$. Since $f_{n_0} * u_\alpha \in I$ (I is an ideal), this completes the proof.

Remark. From this theorem we can draw some immediate conclusions about the ideal theory for $A_p(G)$ [1, Chapter 7]:

a) The closed ideals of $A_p(G)$ are precisely the closed, translation invariant subspaces.

b) Wiener's theorem holds in $A_p(G)$; that is, if $f \in A_p(G)$ and $\hat{f}(y) \neq 0$ for all $y \in \hat{G}$, then the space spanned by the translates of f is norm-dense in $A_p(G)$.

c) In general, spectral synthesis fails in $A_p(G)$.

5. MULTIPLIERS OF $A_p(G)$

A multiplier of $A_p(G)$ is a bounded function ϕ defined on $\hat{G}$ such that $\phi\hat{f}$ is a Fourier transform of some function in $A_p(G)$ wherever $\hat{f}$ is such a Fourier transform. Multipliers of $L_1(G)$ have been defined similarly, and Helson and Edwards proved that the multipliers of $L_1(G)$ are precisely the Fourier-Stieltjes transforms of M(G), the set of finite, complex-valued, regular Borel measures on G [1, p. 73]. In this section we shall show that each Fourier-Stieltjes transform is a multiplier of $A_p(G)$, and that for noncompact groups G each multiplier of $A_p(G)$ is a Fourier-Stieltjes transform.

THEOREM 6. *For each* p $(1 < p < \infty)$ *every Fourier-Stieltjes transform of a measure in* M(G) *is a multiplier of* $A_p(G)$; *and if* G *is noncompact then every multiplier of* $A_p(G)$ *is a Fourier-Stieltjes transform of some measure in* M(G).

Proof. Let $f \in A_p(G)$ and $\mu \in M(G)$. Then $f \in L_1(G)$. Hence $\hat{\mu}\hat{f} \in L_1(G)^\wedge$, where $^\wedge$ denotes the set of Fourier transforms. Since $\hat{\mu}$ is a bounded continuous function and $\hat{f} \in L_p(\hat{G})$, we see that $\hat{\mu}\hat{f} \in L_p(\hat{G})$. Hence $\hat{\mu}\hat{f}$ belongs to $A_p(G)$, and $\hat{\mu}$ is a multiplier.

Now let G be a noncompact group, and suppose ϕ is a multiplier of $A_p(G)$. It is easy to verify that $A_p(G)^\wedge$ is a Banach space with the norm $\|\hat{f}\| = \|f\|^p$ ($\hat{f} \in A_p(G)^\wedge$), and that $B(\hat{G})$, the space of Fourier-Stieltjes transforms of measure in M(G), is a Banach space with the norm $\|\hat{\mu}\|_B = \|\mu\|$ ($\hat{\mu} \in B(\hat{G})$), where $\|\mu\|$ is the total variation of the measure μ. Define the linear transformation $T: A_p(G)^\wedge \to B(\hat{G})$ by $T\hat{f} = \phi\hat{f}$ ($\hat{f} \in A_p(G)^\wedge$). If

$$\hat{f}_n \to \hat{f} \text{ in } A_p(G)^\wedge \quad \text{and} \quad T\hat{f}_n \to \hat{\mu} \text{ in } B(\hat{G}),$$

then $\hat{f}_n \to \hat{f}$ pointwise, and $T\hat{f}_n = \phi\hat{f}_n \to \hat{\mu}$ pointwise. This shows that $\hat{\mu} = \phi\hat{f} = T\hat{f}$, and therefore the transformation T is closed. By the closed-graph theorem we see that T is a bounded linear transformation, and hence there exists some constant K such that

$$\|\phi\hat{f}\|_B \leq K\|\hat{f}\| = K(\|f\|_1 + \|\hat{f}\|_p) \quad (\hat{f} \in A_p(G)^\wedge).$$

Let V be any open subset of $\hat{G}$ with compact closure, and choose f in $A_p(G)$ such that $\hat{f}(y) = 1$ ($y \in V$) [1, p. 48]. If $\hat{\mu} = \phi\hat{f}$, then ϕ is continuous on V, and hence ϕ is continuous on $\hat{G}$.

Given $y_1, y_2, \cdots, y_n$ in $\hat{G}$ and $\varepsilon > 0$, let V be an open subset of G such that $y_i \in V$ ($i = 1, 2, \cdots, n$) and $m(V) < 1$. Such a V exists, since G is noncompact and hence $m\{y_i\} = 0$ ($i = 1, 2, \cdots, n$). Then choose $f \in A_p(G)$ such that

i) $\hat{f}(y_i) = 1$ ($i = 1, 2, \cdots, n$),

ii) $\|f\|_1 < 1 + \varepsilon$, and

iii) $\hat{f}$ has compact support in V.

It follows immediately from the choice of V and f that $\|\hat{f}\|_p < 1 + \varepsilon$.

We see that if $c_1, c_2, \cdots, c_n$ are complex numbers and $\hat{\mu} = \phi\hat{f}$, then

$$\left|\sum_{i=1}^{n} c_i \phi(y_i)\right| = \left|\sum_{i=1}^{n} c_i \phi(y_i)\hat{f}(y_i)\right|$$

$$= \left|\sum_{i=1}^{n} c_i \hat{\mu}(y_i)\right| \leq \|\mu\| \left\|\sum_{i=1}^{n} c_i(\cdot, y_i)\right\|_\infty$$

$$\leq K(\|f\|_1 + \|\hat{f}\|_p)\left\|\sum_{i=1}^{n} c_i(\cdot, y_i)\right\|_\infty$$

$$< 2K(1+\varepsilon)\left\|\sum_{i=1}^{n} c_i(\cdot, y_i)\right\|_\infty .$$

But this last inequality together with the continuity of ϕ implies that ϕ is a Fourier-Stieltjes transform [1, p. 32].

Remark. In general the converse of Theorem 6 for compact groups is not true. Let G be the circle group $\{e^{i\theta} \mid 0 \leq \theta < 2\pi\}$, and let $\{a_n\}_{n=-\infty}^{+\infty}$ be a bounded sequence of complex numbers that is not the Fourier-Stieltjes transform of any measure in M(G). We claim that $\{a_n\}$ is a multiplier of $A_1(G)$. If $f \in A_1(G)$, then $\Sigma_{n=-\infty}^{+\infty} |\hat{f}(n)|$ converges, and so $\Sigma_{n=-\infty}^{+\infty} |a_n \hat{f}(n)|$ also converges. But $\{a_n \hat{f}(n)\}$ is the set of Fourier coefficients of $\Sigma_{n=-\infty}^{+\infty} a_n \hat{f}(n) e^{in\theta}$, which belongs to $A_1(G)$.

In the following, we write $f_z(z) = f(x - z)$ for a function f defined on G, with $x, z \in G$.

THEOREM 7. *If* G *is a noncompact group and* $T: A_p(G) \to A_p(G)$ *is a bounded linear transformation satisfying* $T(f_z) = (Tf)_z$ *for all* $z \in G$, *then there exists some* $\mu \in M(G)$ *such that* $Tf = \mu * f$.

Proof. First we shall show that $(Tf) * g = T(f * g)$ for all $f, g \in A_p(G)$. This will be done by showing that every bounded linear functional on $A_p(G)$ has the same value on both $T(f * g)$ and $(Tf) * g$.

Let F be a bounded linear functional on $A_p(G)$. Then $F \circ T$ is again a bounded linear functional on $A_p(G)$, and there exist functions $\alpha, a \in L_\infty(G)$ and $\beta, b \in L_q(\hat{G})$ $(1/p + 1/q = 1)$ such that

$$F(f) = \int_G f(x)\alpha(x)\,dx + \int_{\hat{G}} \hat{f}(y)\beta(y)\,dy$$

and

$$F \circ T(f) = \int_G f(x)a(x)\,dx + \int_{\hat{G}} \hat{f}(y)b(y)\,dy .$$

From the last relations it follows that

$$\begin{aligned}
F(Tf * g) &= \int_G (Tf * g)(x)\,\alpha(x)\,dx + \int_{\hat{G}} (Tf)^{\wedge}(y)\,\hat{g}(y)\,\beta(y)\,dy \\
&= \int_G \left(\int_G (Tf)(x - z)\,g(z)\,dz\right)\alpha(x)\,dx \\
&\qquad + \int_{\hat{G}} (Tf)^{\wedge}(y)\left(\int_G (-z, y)\,g(z)\,dz\right)\beta(y)\,dy \\
&= \int_G g(z)\left[\int_G T(f_z)(x)\,\alpha(x)\,dx + \int_{\hat{G}} [T(f_z)]^{\wedge}(y)\,\beta(y)\,dy\right] dz \\
&= \int_G g(z)\,F \circ T(f_z)\,dz \\
&= \int_G g(z)\left[\int_G f_z(x)\,a(x)\,dx + \int_{\hat{G}} (f_z)^{\wedge}(y)\,b(y)\,dy\right] dz \\
&= \int_G (f * g)(x)\,a(x)\,dx + \int_{\hat{G}} (f * g)^{\wedge}(y)\,b(y)\,dy \\
&= F \circ T(f * g).
\end{aligned}$$

Hence $Tf * g = T(f * g)$, and by symmetry, $Tf * g = Tg * f$. Thus $(Tf)^{\wedge}\hat{g} = (Tg)^{\wedge}\hat{f}$. From this it follows that there exists a function ϕ on $\hat{G}$ such that $(Tf)^{\wedge} = \phi\hat{f}$ for all $f \in A_p(G)$. Clearly, ϕ is a multiplier and has the form $\hat{\mu}$ with $\mu \in M(G)$, by Theorem 6. Thus $(Tf)^{\wedge} = \hat{\mu}\,\hat{f}$, and therefore, by the uniqueness of the Fourier-Stieltjes transforms, $Tf = \mu * f$.

6. $A_2(G)$

With the results of the preceding sections it is simple to prove the following theorem.

THEOREM 8. $L_1(G) \cap L_2(G) = A_2(G)$.

Proof. We are considering $L_1(G) \cap L_2(G)$ as a Banach space with the norm

$$\|f\|_{1,2} = \|f\|_1 + \|f\|_2 \qquad (f \in L_1(G) \cap L_2(G)).$$

By the Plancherel theorem it is clear that we can consider $L_1(G) \cap L_2(G)$ as a closed subspace of $A_2(G)$, and it is easy to verify that $L_1(G) \cap L_2(G)$ is an ideal in $A_2(G)$.

However, if I_1 denotes the closure of $L_1(G) \cap L_2(G)$ in $L_1(G)$, then from Theorem 5 and the relation $I_1 = L_1(G)$ we conclude that

$$L_1(G) \cap L_2(G) = I_1 \cap A_2(G) = A_2(G).$$

Remarks. a) It follows immediately from the theorem that if $f \in A_2(G)$ and g is the inverse Plancherel transform of $\hat{f}$, then $f = g$.

b) The plausible conjecture that $L_1(G) \cap L_p(G) = A_q(G)$ $(1 < p < 2,\ 1/p + 1/q = 1)$ is false.

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