CONCERNING THE ORDER STRUCTURE OF KÖTHE SEQUENCE SPACES, II

A. L. Peressini

1. INTRODUCTION

This paper investigates the order structure of the space $L(\lambda, \mu)$ of weakly continuous linear mappings of a sequence space λ into another sequence space μ. The weak topologies referred to here are those formed with respect to the respective α-duals of λ and μ, while the order structure of $L(\lambda, \mu)$ is that generated by the positive mappings in $L(\lambda, \mu)$ when λ and μ are equipped with their natural order. We shall use several important results concerning the algebraic structure of $L(\lambda, \mu)$ that are found in the fundamental paper [6] of G. Köthe and O. Toeplitz and in the work of H. S. Allen (see [1] and Chapter 6 of [4]). We shall also use the results and terminology of our earlier work [8].

2. PRELIMINARY MATERIAL

Throughout this paper we shall assume that λ and μ are real sequence spaces containing the space ϕ of sequences with only a finite number of nonzero components. The positive cones of sequences with nonnegative components in λ and μ will be denoted systematically by K_λ and K_μ, respectively; K'_λ and K'_μ will denote the corresponding dual cones in the α-duals λ^* and μ^* of λ and μ, respectively. We shall always assume that λ is a solid; that is, if $|x| \leq |y|$ and $y \in \lambda$, then $x \in \lambda$ (here $|x| = (|x_i|)$ denotes the lattice-theoretic absolute value of x in λ). We refer the reader to [5] and [8] for further details concerning the topological and order-theoretic properties of sequence spaces.

A *matrix transformation* on λ into μ is an infinite matrix $A = (a_{ij})$ with the following properties:

(M_1) For each $x = (x_j) \in \lambda$, the series $\Sigma_{j=1}^{\infty} a_{ij}x_j$ converges absolutely for each i.

(M_2) For each $x = (x_j) \in \lambda$, the equation $y_i = \Sigma_{j=1}^{\infty} a_{ij}x_j$ defines an element $y = (y_i)$ of μ.

If $A = (a_{ij})$ is a matrix transformation on λ into μ, then the mapping $y = Ax$ defined by (M_2) is clearly a linear mapping of λ into μ. On the other hand, if T is a linear mapping of λ into μ and if there exists a matrix transformation A of λ into μ such that $Tx = Ax$ for all $x \in \lambda$, then T is *represented* by A. If T is represented by a matrix transformation A, then A is unique, since λ contains the "unit vectors" $e^{(k)} = (\delta_{ik}: i = 1, 2, \cdots)$ (δ_{ik} denotes the Kronecker delta). The following result, essentially due to G. Köthe, O. Toeplitz, and H. S. Allen, is stated here in a form convenient for our purposes.

(2.1) PROPOSITION. *The following conditions on a linear mapping* T *of* λ *into* μ *are equivalent:*

Received February 10, 1964.
This work was supported in part by a National Science Foundation contract.

a) T *is represented by a matrix transformation of* λ *into* μ.

b) T *is sequentially continuous for the weak topologies* $\sigma(\lambda, \lambda^*)$ *and* $\sigma(\mu, \mu^*)$.

c) T *is continuous for* $\sigma(\lambda, \lambda^*)$ *and* $\sigma(\mu, \mu^*)$.

If T *is continuous for* $\sigma(\lambda, \lambda^*)$ *and* $\sigma(\mu, \mu^*)$, *then the adjoint mapping* T' *is represented by the transpose* A^T *of the matrix transformation* A *representing* T.

Proof. a) implies c) by (6.2, II) of [4]. c) obviously implies b), and b) implies a) by Satz 1 of Section 8 in [6]. The last assertion also follows from (6.2, II) of [4].

3. THE SPACES $L^b(\lambda, \mu)$ AND $L(\lambda, \mu)$

In accordance with accepted usage, a linear mapping T of λ into μ is *positive* (respectively, *order-bounded*) if $Tx \geq \theta$ whenever $x \geq \theta$ (respectively, if T maps every order-bounded set in λ into an order-bounded set in μ). The vector space $L^b(\lambda, \mu)$ of all order-bounded linear mappings of λ into μ is ordered by the cone $\mathfrak{N}_b$ of all positive linear mappings of λ into μ. It is easy to verify that if $\mathfrak{N} = \mathfrak{N}_b \cap L(\lambda, \mu)$, then $T \in \mathfrak{N}$ if and only if T is represented by a matrix A whose entries are all nonnegative. In our discussions of order properties of $L^b(\lambda, \mu)$ and $L(\lambda, \mu)$, the order structure shall be understood to be that generated by $\mathfrak{N}_b$ and $\mathfrak{N}$, respectively.

The following generalizes a known result concerning linear functionals (see for example [3, pp. 35-36]; the formulas on p. 36 also carry over to this more general context).

(3.1) PROPOSITION. *If* μ *is a solid sequence space, then* $L^b(\lambda, \mu)$ *is an order-complete vector lattice.*

Proof. If $T \in L^b(\lambda, \mu)$ and $x \in K_\lambda$, then $T[\theta, x]$ is majorized in μ. Define $T^+(x) = \sup\{Ty: y \in [\theta, x]\}$; then T^+ is clearly positively homogeneous on K_λ; moreover, T^+ is additive on K_λ, since λ is a lattice. If we define

$$T^+ x = T^+ x^+ - T^+ x^-$$

for each $x \in \lambda$, then T^+ is a positive linear mapping of λ into μ, and we can easily verify that T^+ is the supremum of T and 0 in $L^b(\lambda, \mu)$. If $\mathfrak{M}$ is a directed($\leq$) subset (that is, if for all T_1, T_2 in $\mathfrak{M}$, there is a $T_3 \in \mathfrak{M}$ such that $T_3 \geq T_1$, $T_3 \geq T_2$) of $\mathfrak{N}_b$ that is majorized by $T_0 \in \mathfrak{N}_b$, then the mapping T_1 of K_λ into μ defined by

$$T_1 x = \sup\{Tx: T \in \mathfrak{M}\}$$

is additive since $\mathfrak{M}$ is directed($\leq$), and it is clearly positively homogeneous. Therefore the extension of T_1 to λ is a positive linear map that is the supremum of $\mathfrak{M}$. We conclude that $L^b(\lambda, \mu)$ is an order-complete vector lattice.

Even if λ and μ are perfect, it is not generally true that $L(\lambda, \mu)$ is a lattice. For example, suppose that $\lambda = \mu = \ell^2$ and that $A = (a_{ij})$ is defined by

$$a_{ij} = \begin{cases} \dfrac{1}{i-j} & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases}$$

Then A satisfies Hilbert's criterion for elements of $L(\ell^2, \ell^2)$, while $|A|$ does not.

(3.2) PROPOSITION. *If* $L(\lambda, \mu)$ *is a lattice, then* μ *is a lattice and* $L(\lambda, \mu)$ *coincides with the space of linear mappings on* λ *into* μ *that are continuous for* $o(\lambda, \lambda^*)$ *and* $o(\mu, \mu^*)$. *Moreover, if* μ *is solid, then* $L(\lambda, \mu)$ *is a solid sublattice of* $L^b(\lambda, \mu)$.

Proof. By (6.4, I) in [4], the column space of $L(\lambda, \mu)$ is μ; hence, if $L(\lambda, \mu)$ is a lattice, then μ is a lattice. Suppose $L(\lambda, \mu)$ is a lattice; then, in particular, $L(\lambda, \mu) = \mathfrak{N} - \mathfrak{N}$, that is, $\mathfrak{N}$ is a generating cone in $L(\lambda, \mu)$.

We assert that whenever $\mathfrak{N}$ is generating, then $L(\lambda, \mu)$ is precisely the class of linear maps on λ into μ that are continuous for $o(\lambda, \lambda^*)$ and $o(\mu, \mu^*)$. For if $T \in L(\lambda, \mu)$, then $T = T_1 - T_2$, where T_1 and T_2 are positive, weakly continuous linear maps. It follows that the adjoints T_1' and T_2' are positive mappings that are continuous for $\sigma(\mu^*, \mu)$ and $\sigma(\lambda^*, \lambda)$. By Theorem (4.1) in [7] we conclude that T_1 and T_2 are continuous for $o(\lambda, \lambda^*)$ and $o(\mu, \mu^*)$; hence T is continuous for these topologies. On the other hand, if T is continuous for the normal topologies $o(\lambda, \lambda^*)$ and $o(\mu, \mu^*)$, then T is continuous for $\sigma(\lambda, \lambda^*)$ and $\sigma(\mu, \mu^*)$, since the normal topologies are coarser than the respective Mackey topologies (see Section 30, 2(4) in [5]). It now follows from (2.1) that $L(\lambda, \mu)$ is the space of linear mappings on λ into μ that are continuous for $o(\lambda, \lambda^*)$ and $o(\mu, \mu^*)$.

Suppose that μ is solid, that $A \in L^b(\lambda, \mu)$, and that $|A| \leq |B|$ for some $B \in L(\lambda, \mu)$. If $x_\alpha \to \theta$ in λ for $o(\lambda, \lambda^*)$, then $|x_\alpha| \to \theta$ for $o(\lambda, \lambda^*)$, since λ is a locally convex lattice for this topology. Since $|B| \in L(\lambda, \mu)$, we conclude that $|B|\,|x_\alpha| \to \theta$ for $o(\mu, \mu^*)$; hence $|A|\,|x_\alpha| \to \theta$ for $o(\mu, \mu^*)$, since K_μ is a normal cone for $o(\mu, \mu^*)$. Because $|Ax_\alpha| \leq |A|\,|x_\alpha|$, it follows that $Ax_\alpha \to \theta$ for $o(\mu, \mu^*)$, since μ is a locally convex lattice for this topology. Therefore $A \in L(\lambda, \mu)$, that is, $L(\lambda, \mu)$ is a solid sublattice of $L^b(\lambda, \mu)$.

The method employed to prove the last part of Proposition (3.2) can easily be modified to establish the following result: $L(\lambda, \mu)$ is a lattice if μ is solid and $\mathfrak{N}$ is generating.

The next result gives a number of sufficient conditions for $L(\lambda, \mu)$ to be a lattice.

(3.3) PROPOSITION. *Each of the following conditions implies that* $L(\lambda, \mu)$ *is a lattice:*

a) λ *is perfect and* μ *is either the space* (c) *of convergent sequences or the space* (c_0) *of null-sequences.*

b) λ *is perfect and* μ *is the space* (m) *of bounded sequences.*

c) μ *is perfect and* $\lambda = \ell^1$.

d) $\lambda \dashv \phi$ *and* μ *is perfect.*

Proof. a) Suppose that $A = (a_{ij}) \in L(\lambda, (c))$, and define $\phi^{(i)} = (a_{ij}: j = 1, 2, \cdots)$. Then $\phi^{(i)} \in \lambda^*$ by (M_1), and if $x \in \lambda$, then $\{\langle x, \phi^{(i)}\rangle: i = 1, 2, \cdots\}$ is an element of (c) by (M_2). Therefore $\{\phi^{(i)}\}$ is a Cauchy sequence for $\sigma(\lambda^*, \lambda)$, hence $\{\phi^{(i)}\}$ is convergent since λ is perfect. The lattice operations in λ^* are $\sigma(\lambda^*, \lambda)$-sequentially continuous by Proposition 2 of [8], hence $\{|\phi^{(i)}|\}$ is $\sigma(\lambda^*, \lambda)$-convergent. For a given $x \in \lambda$, define $y = (y_i)$ by

$$y_i = \sum_{j=1}^{\infty} |a_{ij}|\, x_j = \langle x, |\phi^{(i)}|\rangle ;$$

then $y \in (c)$, and since $|\phi^{(i)}| \in \lambda^*$, the series is absolutely convergent for each i. It follows that $|A| \in L(\lambda, (c))$. A similar proof shows that $L(\lambda, (c_0))$ is a lattice.

b) Suppose that $A \in L(\lambda, (m))$, and define $\{\phi^{(i)}\}$ as in a); then $\phi^{(i)} \in \lambda^*$ for each i, and the sequence $\{\langle x, \phi^{(i)}\rangle: i = 1, 2, \cdots\}$ is in (m) for each $x \in \lambda$. Therefore $\{\phi^{(i)}\}$ is $\sigma(\lambda^*, \lambda)$-bounded. It follows from Section 30, 5(6) in [5] that $\{|\phi^{(i)}|\}$ is $\sigma(\lambda^*, \lambda)$-bounded, since λ is perfect. Thus $\langle x, |\phi^{(i)}|\rangle = \Sigma_{j=1}^{\infty} |a_{ij}| x_j$ is absolutely convergent for each i and each $x \in \lambda$; moreover, the sequence $y = (y_i)$ defined by

$$y_i = \sum_{j=1}^{\infty} |a_{ij}| x_j = \langle x, |\phi^{(i)}|\rangle$$

is in (m) for each $x \in \lambda$. Therefore $|A| \in L(\lambda, (m))$.

c) Since both ℓ^1 and μ are perfect, we know that

$$L(\ell^1, \mu)^T = \{A^T: A \in L(\ell^1, \mu)\} = L(\mu^*, (m))$$

(see (6.4, III) in [4]). Therefore, if $A \in L(\ell^1, \mu)$, then $A^T \in L(\mu^*, (m))$. By b), $|A|^T = |A^T| \in L(\mu^*, (m))$, hence $|A| \in L(\ell^1, \mu)$, that is, $L(\ell^1, \mu)$ is a lattice.

d) Obviously, if $\mu = \omega$, then $L(\lambda, \mu)$ is a lattice. But since μ is perfect, we see that $L(\phi, \mu) = L(\mu^*, \omega)^T$ by (6.4, II) in [4], and hence $L(\phi, \mu)$ is a lattice.

Since λ is solid, the cone K_λ in λ is generating, that is, $\lambda = K_\lambda - K_\lambda$. Hence the class $\mathfrak{S} = \{[-x, x]: x \in K_\lambda\}$ is a fundamental system of order-bounded sets in λ. Each order interval in a sequence space is bounded for the weak topology formed with respect to the α-dual of the space, since the dual cone is generating. It follows easily from this that the $\mathfrak{S}$-topology on $L^b(\lambda, \mu)$ or $L(\lambda, \mu)$ (that is, the topology of uniform convergence on order-bounded sets) is locally convex whenever μ is equipped with any locally convex topology $\mathfrak{T}$ such that $\mu(\mathfrak{T})' = \mu^*$. In particular, if $\mathfrak{T} = \sigma(\mu, \mu^*)$ (respectively, $\mathfrak{T} = o(\mu, \mu^*)$), we shall refer to the corresponding $\mathfrak{S}$-topology on $L^b(\lambda, \mu)$ or $L(\lambda, \mu)$ as the $\mathfrak{S}_\sigma$-topology (respectively, the $\mathfrak{S}_o$-topology).

(3.4) PROPOSITION. *The cones* $\mathfrak{R}_b$ *and* $\mathfrak{R}$ *are closed and normal in* $L^b(\lambda, \mu)$ *and* $L(\lambda, \mu)$, *respectively, for the* $\mathfrak{S}_\sigma$*-topology and the* $\mathfrak{S}_o$*-topology.*

Proof. The fact that $\mathfrak{R}$ and $\mathfrak{R}_b$ are closed cones follows as in (8.1) of [9]. If $x \in K_\lambda$, then $[-x, x] = [\theta, x] - [\theta, x]$, hence the class $\{[\theta, x] - [\theta, x]: x \in K_\lambda\}$ is a fundamental system of order-bounded subsets of λ. Therefore $\mathfrak{R}_b$ and $\mathfrak{R}$ are normal for the $\mathfrak{S}_\sigma$-topology and the $\mathfrak{S}_o$-topology since K_μ is normal for $o(\mu, \mu^*)$ and $\sigma(\mu, \mu^*)$.

(3.5) PROPOSITION. *If* μ *is solid, then the* $\mathfrak{S}_o$*-topology on* $L^b(\lambda, \mu)$ *(respectively, on* $L(\lambda, \mu)$ *if* $L(\lambda, \mu)$ *is a lattice) is the coarsest topology* $\mathfrak{T}$ *finer than the topology of simple convergence for which the lattice operations are* $\mathfrak{T}$*-continuous.*

Proof. Let $N_0 = \{T: T([-x_0, x_0]) \subset [-v_0, v_0]^0\}$ be an $\mathfrak{S}_o$-neighborhood of θ, where $x_0 \in K_\lambda$ and $v_0 \in K'_\mu$. Then $N_s = \{T: T(x_0) \in [-v_0, v_0]^0\}$ is a θ-neighborhood for $\mathfrak{T}$. If the lattice operations are $\mathfrak{T}$-continuous, there exists a $\mathfrak{T}$-neighborhood V of θ such that $|T| \in N_s$ whenever $T \in V$. Then, if $x \in [-x_0, x_0]$, $T \in V$, and $v \in [-v_0, v_0]$, we see that

$$\begin{aligned}\langle Tx, v\rangle &= \langle T^+x^+, v\rangle - \langle T^+x^-, v\rangle - \langle T^-x^+, v\rangle + \langle T^-x^-, v\rangle \\ &\leq \langle T^+x^+, v_0\rangle + \langle T^+x^-, v_0\rangle + \langle T^-x^+, v_0\rangle + \langle T^-x^-, v_0\rangle \\ &= \langle T^+|x|, v_0\rangle + \langle T^-|x|, v_0\rangle = \langle |T|\,|x|, v_0\rangle \leq \langle |T|x_0, v_0\rangle \leq 1.\end{aligned}$$

Therefore $T \in N_0$, that is, $\mathfrak{T}$ is finer than the $\mathfrak{S}_o$-topology.

To complete the proof, it is enough to prove the continuity of the lattice operations at θ for the $\mathfrak{S}_o$-topology, since $\mathfrak{N}_b$ and $\mathfrak{N}$ are normal for this topology by Proposition (3.4). Given a θ-neighborhood N_0 (defined as in the first part of the proof), $x \in [-x_0, x_0]$, and $T \in N_0$, we note that

$$|T|x \leq |T|\,x_0 = \sup\{Tz\colon z \in [-x_0, x_0]\}.$$

Now $Tz \in [-v_0, v_0]^0$ for each $z \in [-x_0, x_0]$, since $T \in N_0$; hence, since $o(\mu, \mu^*)$ is locally order-complete by Proposition (3.2) in [7], we conclude that

$$|T|x \in [-v_0, v_0]^0,$$

that is, $|T| \in N_0$. Therefore the lattice operations are continuous at θ for the $\mathfrak{S}_o$-topology.

(3.6) PROPOSITION. *If λ and μ^* have order units, the $\mathfrak{S}_o$-topology is normable. If the $\mathfrak{S}_o$-topology is normable, then λ has an order unit.*

Proof. If x_0 and v_0 are order units in λ and μ^*, respectively, then the class $\{n[-x_0, x_0]\colon n = 1, 2, \cdots\}$ is a fundamental system of order-bounded sets in λ, the family $\left\{\frac{1}{n}[-v_0, v_0]^0\colon n = 1, 2, \cdots\right\}$ is a neighborhood basis in μ for $o(\mu, \mu^*)$, and hence the positive multiples of the set

$$N_0 = \{T\colon T([-x_0, x_0]) \subset [-v_0, v_0]^0\}$$

form a neighborhood basis of θ for the $\mathfrak{S}_o$-topology. Therefore the $\mathfrak{S}_o$-topology is normable.

Suppose that the $\mathfrak{S}_o$-topology is normable and that U is the unit ball for this topology. Then for some $x_0 \in K_\lambda$ and $v_0 \in K'_\mu$, the neighborhood N_0 defined in the first part of the proof is contained in U. Therefore, if $x \in K_\lambda$, there exists a $\lambda_0 > 0$ such that $\frac{1}{\lambda_0}T([-x_0, x_0]) \subset [-v_0, v_0]^0$ implies that $Tx \in [-v_0, v_0]^0$, since $\{x\}$ is order-bounded. If $x \notin \left[-\frac{1}{\lambda_0}x_0, \frac{1}{\lambda_0}x_0\right]$, there exists a $\sigma(\lambda, \lambda^*)$-continuous linear functional f_0 on λ such that

$$f_0(x) > 1 > \sup\left\{f_0(y)\colon y \in \left[-\frac{1}{\lambda_0}x_0, \frac{1}{\lambda_0}x_0\right]\right\}.$$

Choose $w_0 \in [-v_0, v_0]^0$ so that $|\langle w_0, v_1\rangle| = 1$ for some $v_1 \in [-v_0, v_0]$. The linear mapping T_0 of λ into μ defined by

$$T_0 z = f_0(z)\, w_0 \qquad (z \in \lambda)$$

is obviously continuous for $\sigma(\lambda, \lambda^*)$ and $\sigma(\mu, \mu^*)$. Therefore

$$|\langle T_0 y, v\rangle| = |f_0(y)|\,|\langle w_0, v\rangle| \leq 1$$

for all $y \in \left[-\frac{1}{\lambda_0}x_0, \frac{1}{\lambda_0}x_0\right]$ and for all $v \in [-v_0, v_0]$; hence, by virtue of the choice of λ_0, we conclude that $Tx \in [-v_0, v_0]^0$. However, by the definition of T_0,

$$|\langle T_0 x, v_1\rangle| = |f_0(x)|\,|\langle w_0, v_1\rangle| > 1,$$

and hence $T_0 x \notin [-v_0, v_0]^0$. This contradiction proves that $x \in \left[-\frac{1}{\lambda_0}x_0, \frac{1}{\lambda_0}x_0\right]$, that is, x_0 is an order unit of λ.

It is not generally true that the supremum of a directed($\leq$) subset of an ordered, locally convex space is in the closure of the subset. (For example, the set

$$\left\{u^{(k)} = \sum_{n=1}^{k} e^{(n)} : k = 1, 2, \cdots\right\}$$

has the supremum $e = (1, 1, 1, \cdots)$ in the space (m) of bounded real sequences, yet the sphere of radius 1/2 centered at e does not contain $u^{(k)}$ for any k.) However, we can prove the following result.

(3.7) LEMMA. *If* $\mathfrak{M}$ *is a* directed($\leq$) *(respectively,* directed($\geq$)) *subset of a solid sequence space* μ *that has a supremum* a *(respectively, infimum* a), *then the filter* $\mathfrak{F}(\mathfrak{M})$ *of sections of* $\mathfrak{M}$ *converges to* a *for* $o(\mu, \mu^*)$.

Proof. Since μ is a solid sequence space, μ is an order-complete sublattice of μ^{**}, hence a is also the supremum (respectively, infimum) of $\mathfrak{M}$ in μ^{**}. Without loss in generality, we can assume that $a = \theta$ and that $\mathfrak{M}$ is directed($\geq$).

In view of the fact that $o(\mu^{**}, \mu^*)$ induces $o(\mu, \mu^*)$ on μ, it would suffice to show that the filter base $\mathfrak{F}(\mathfrak{M})$ converges to θ for $o(\mu^{**}, \mu^*)$. Since μ^{**} is perfect, it is complete for $o(\mu^{**}, \mu^*)$ by Section 30, 5(7) in [5]. Therefore, if we can show that $\mathfrak{F}(\mathfrak{M})$ is a Cauchy filter for $o(\mu, \mu^*)$, it would follow from Proposition 6 on p. 26 of [3] that $\mathfrak{F}(\mathfrak{M})$ converges to θ. Corresponding to an $o(\mu^{**}, \mu^*)$-neighborhood $[-u_0, u_0]^0$ ($u_0 \in K_\mu'$) of θ and a section S in $\mathfrak{F}(\mathfrak{M})$, define

$$\alpha_0 = \inf\{\langle y, u_0\rangle : y \in S\}.$$

Choose $y_0 \in S$ so that $\langle y, u_0\rangle - \alpha_0 \leq 1/2$ for all $y \in \mathfrak{M}$ with $y \leq y_0$. If $y_1 \leq y_0$, $y_2 \leq y_0$, and $v \in [-u_0, u_0]$, then

$$\begin{aligned}|\langle v, y_1 - y_2\rangle| &\leq |\langle v, y_0 - y_1\rangle| + |\langle v, y_0 - y_2\rangle| \\ &\leq \langle u_0, y_0 - y_1\rangle + \langle u_0, y_0 - y_2\rangle \\ &\leq \langle u_0, y_0\rangle - \alpha_0 + \langle u_0, y_0\rangle - \alpha_0 \leq 1.\end{aligned}$$

Therefore, the section $S_0 = \{y \in \mathfrak{M} : y \leq y_0\}$ satisfies the relation

$$S_0 - S_0 \subset [-u_0, u_0]^0,$$

that is, $\mathfrak{F}(\mathfrak{M})$ is a Cauchy filter.

Recall that a solid subspace B of an order-complete vector lattice E is a *band* in E if B contains the supremum of every subset of B that is majorized in E (see [3], Chapter 2, Section 1, No. 5). Also, a vector lattice E is *locally order-complete* for a locally convex topology $\mathfrak{T}$ on E if there exists a $\mathfrak{T}$-neighborhood basis of θ consisting of convex, solid, order-complete sets.

(3.8) PROPOSITION. *If* μ *is a solid sequence space and* L *is a sublattice of* $L^b(\lambda, \mu)$ *that is order-complete, then* L *is locally order-complete for the induced* $\mathfrak{S}_0$*-topology.* B *is a band in* L *if and only if* B *is an* $\mathfrak{S}_0$*-closed solid subspace of* L.

Proof. Suppose $\mathfrak{M}$ is a majorized directed($\leq$) subset of $\mathfrak{K}_b \cap L$, and define $T_0 = \sup \mathfrak{M}$. Let

$$N_0 = \{T \in L: T[-x_0, x_0] \subset [-u_0, u_0]^0\} \qquad (x_0 \in K_\lambda, u_0 \in K'_\mu)$$

be any $\mathfrak{S}_0$-neighborhood of θ in L. Since $T_0 x_0 = \sup \{Tx_0: T \in \mathfrak{M}\}$, it follows from Lemma (3.7) that there exists a $T_1 \in \mathfrak{M}$ such that $(T_0 - T_1)(x_0) \in [-u_0, u_0]^0$. Thus, if $x \in [-x_0, x_0]$, $T \geq T_1$, and $T \in \mathfrak{M}$, then

$$|(T_0 - T)(x)| \leq (T_0 - T)(x_0) \leq (T_0 - T_1)(x_0);$$

hence, since $[-u_0, u_0]^0$ is solid, we conclude that

$$(T_0 - T)[-x_0, x_0] \subset [-u_0, u_0]^0,$$

that is, $T \in T_0 + N_0$ for all $T \in \mathfrak{M}$ such that $T \geq T_1$. Therefore the filter $\mathfrak{F}(\mathfrak{M})$ of sections of $\mathfrak{M}$ converges to sup $\mathfrak{M}$ for the induced $\mathfrak{S}_0$-topology on L. It follows that there exists a neighborhood basis of θ in L for the induced $\mathfrak{S}_0$-topology consisting of solid, order-complete sets; that is, L is locally order-complete for this topology.

Suppose B is an $\mathfrak{S}_0$-closed solid sublattice of L, and let $\mathfrak{M}$ be a directed($\leq$) subset of L that is majorized in L; then sup $\mathfrak{M} \in \overline{B} = B$, by the first part of the proof; that is, B is a band in L.

Conversely, suppose B is a band in L; then B is certainly a solid subspace of L, and L is the order-direct sum of B and the $B^\perp$ of elements disjoint from B (see [3, p. 25, Theorem 1]). If P denotes the projection of L onto B, then $|Pz| = P(|z|) \leq |z|$; hence P is continuous for the induced $\mathfrak{S}_0$-topology by Propositions (3.4) and (3.5). Since $B = (I - P)^{-1}(\theta)$, where I denotes the identity map on L, it follows that B is closed for the induced $\mathfrak{S}_0$-topology.

REFERENCES

1. H. S. Allen, *Maximum matrix rings*, J. London Math. Soc. 18 (1943), 142-147.
2. G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloquium Publications, Vol. 25, revised edition, Amer. Math. Soc., New York, 1948.
3. N. Bourbaki, *Eléments de mathématique, Livre VI, Intégration*, Actualités Sci. Ind., no. 1175, Hermann et Cie, Paris, 1952.
4. R. G. Cooke, *Linear Operators. Spectral theory and some other applications.* Macmillan, London, 1953.

5. G. Köthe, *Topologische lineare Räume, I,* Die Grundlehren der mathematischen Wissenschaften, Bd. 107, Springer-Verlag, Berlin, 1960.

6. G. Köthe and O. Toeplitz, *Lineare Räume mit unendlichvielen Koordinaten und Ringe unendlicher Matrizen,* J. Reine Angew. Math. 171 (1934), 193-226.

7. A. L. Peressini, *On topologies in ordered vector spaces,* Math. Ann. 144 (1961), 199-223.

8. ———, *Concerning the order structure of Köthe sequence spaces,* Michigan Math. J. 10 (1963), 409-415.

9. H. Schaefer, *Halbgeordnete lokalkonvexe Vektorräume, II,* Math. Ann. 138 (1959), 259-286.

University of Illinois