ANALYTIC CONTINUATION AND SUMMABILITY OF POWER SERIES

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Given a domain in which a linear method of summability sums the geometric series $\Sigma_{n=0}^{\infty} z^n$ to $(1 - z)^{-1}$, we shall under certain conditions obtain a set in which the method sums a power series with a positive radius of convergence to one of its analytic continuations. In order to state our main result, we need the following definitions and lemmas.

By Γ we denote a family of Jordan arcs γ in the complete complex plane, with endpoints $0, \infty$, directed from 0 to ∞, and having the following properties: (a) if γ_1 and γ_2 are two different elements of Γ, then they intersect only at 0 and ∞; (b) to each complex z $(z \neq 0, \infty)$ corresponds an element $\gamma(z) = \gamma \in \Gamma$ passing through z. We write $[0, z]$ and $[z, \infty]$ for the subarcs of $\gamma(z)$ with endpoints 0 and z and with endpoints z and ∞, respectively, and we replace a bracket by a parenthesis to indicate that the corresponding endpoint is delated from the subarc.

If A and B are two point sets, we denote by $d(A, B)$ the distance between them,

by AB the set $\{s \mid s = zw,\ z \in A,\ w \in B\}$,

by A^{-1} the set $\{z \mid z^{-1} \in A\}$,

by wA the set $\{s \mid s = wz,\ z \in A\}$,

and by A^c the complement of A relative to the complete complex plane.

A family Γ will be called *continuous* provided to each $z_1 \neq 0, \infty$ and each $\varepsilon > 0$ there corresponds a $\delta = \delta(z_1, \varepsilon) > 0$ such that

$$\sup_{w \in [0,z]} d(w, [0, z_1]) < \varepsilon$$

for all points z in the disk $|z - z_1| < \delta$. The following example shows that an arbitrary family Γ is not necessarily continuous. Let γ_0 be the linear ray $z \geq 0$. For $n \geq 1$, let γ_n be the polygonal line composed of the two line segments

$$[0,\ 3 + 3 \cdot 2^{-2n} i] \quad \text{and} \quad [3 + 3 \cdot 2^{-2n} i,\ 2 + 3 \cdot 2^{-(2n+1)} i]$$

and the ray $t + 3 \cdot 2^{-(2n+1)} i$ $(t \geq 2)$. It is easy to see that we can embed the sequence $\{\gamma_n\}_0^{\infty}$ in a family Γ (not uniquely). Suppose this is done, and choose $z_1 = 2$ and $z = 2 + 2^{-(2n+1)} i$. Then

$$\sup_{w \in [0,z]} d(w, [0, 2]) \geq d(3 + 3 \cdot 2^{-2n} i,\ 2) > 1 .$$

Choosing $\varepsilon = 1$, we see that Γ is not continuous.

Received May 9, 1963 and February 21, 1964.

The research in this paper was sponsored in part by the Air Force Office of Scientific Research, OAR, through the European Office, Aerospace Research, United States Air Force, and through the Hebrew University.

Denote by $P \equiv P(z)$ a power series $\Sigma_0^\infty a_n z^n$ with the partial sums $s_n(z)$ and with a positive radius of convergence. Continue $P(z)$ analytically along each $\gamma \in \Gamma$ from 0 to the first singular point $w(\gamma)$ on γ. If there is no finite singular point on γ, we define $w(\gamma) = \infty$. By $M \equiv M(P; \Gamma)$ we denote the union of all the sets $[0, w(\gamma))$, and we call this set the Γ-*Mittag-Leffler star* of $P(z)$. Clearly, $\infty \notin M$. If $z_0 \in M$, we denote by $P(z_0; \Gamma)$ the value at z_0 of the analytic continuation of $P(z)$ along $\gamma(z_0)$. By definition, $P(z; \Gamma)$ is a single-valued function in M.

A set D is called a Γ-*star set* provided it is not empty, $\infty \notin D$, and $z \in D$ implies $[0, z) \subset D$. A Γ-*star domain* is a Γ-star set that is also a domain. Obviously, a Γ-star domain is simply connected, a union of Γ-star domains is a Γ-star domain, and an intersection of Γ-star sets is a Γ-star set.

For a family Γ, we define the set $D(\Gamma)$ by

$$D(\Gamma) \equiv \{s \mid s = z/w,\ z \neq 0, \infty,\ w \in (0, z]\}.$$

A set D is Γ-*regular* if $0 \in D$, $1 \notin D$, $\infty \notin D$, and $D(\Gamma) \subset D^c$.

LEMMA 1. *If* Γ *is continuous, then* $M(P; \Gamma)$ *is a simply connected domain and* $P(z; \Gamma)$ *is holomorphic in* $M(P; \Gamma)$. *If* Γ *is not continuous, then* $M(P; \Gamma)$ *is not necessarily a domain.*

Proof. We have to show that if Γ is continuous, then $M(P; \Gamma)$ is a simply connected domain and $\frac{d}{dz}P(z; \Gamma)$ exists for all points $z \in M$. If $z_0 \in M$ and $z_0 \neq 0$, then there exists a domain G and a function f, holomorphic in G, such that $[0, z_0] \subset G$ and $f(z) = P(z; \Gamma)$ for $z \in [0, z_0]$. The continuity of Γ implies the existence of a $\delta > 0$ such that $[0, z] \subset G$ whenever $|z - z_0| < \delta$. Therefore

$$\{z \mid |z - z_0| < \delta\} \subset M(P; \Gamma)$$

and $P(z; \Gamma) = f(z)$ for these values of z. Thus $P'(z_0; \Gamma)$ exists and $M(P; \Gamma)$ is an open set. The first part of the lemma now follows from the general properties of Γ-star sets.

Next, let Γ be the noncontinuous family described earlier. In order to prove our theorem it is enough to show the existence of a power series $P(z)$ with a positive radius of convergence such that $M(P; \Gamma)$ is not a domain. Choose

$$a_n = 5/2 + 19 \cdot 2^{-(2n+3)} i, \quad b_n = 2 + 3 \cdot 2^{-(2n+1)} i \qquad (n \geq 1).$$

For the function $\text{Log}\{(z - a_n)/(b_n - a_n)\}$ $(n \geq 1)$, choose at $z = 0$ the branch which, if continued analytically from 0 to b_n along the linear segment $[0, b_n]$, yields at $z = b_n$ the value $\text{Log } 1 = 2\pi i$. The function

$$P(z) = \sum_{n=1}^{\infty} \{n! \text{ Log}[(z - a_n)/(z_n - b_n)]\}^{-1}$$

is holomorphic in $|z| < 5/2$. For the Jordan arc γ_0 of our discontinuous family, $w(\gamma_0) = 5/2$. For the Jordan arcs γ_n, $w(\gamma_n) = b_n$. This means that

$$\{z \mid 0 \leq z < 5/2\} \subset M(P; \Gamma)$$

and

$$\{z \mid z = t + 3\cdot 2^{-(2n+1)}i,\ t \geq 2\} \subset M(P;\ \Gamma)^c \qquad (n \geq 1).$$

Hence each point z $(2 \leq z < 5/2)$ is not an interior point of $M(P;\ \Gamma)$, and $M(P;\ \Gamma)$ is not a domain. This completes the proof of Lemma 1.

LEMMA 2. *Let* D *be a* Γ*-regular set. Suppose* γ *is a bounded Jordan curve whose interior contains the point* 0. *If a point set* F *satisfies the condition*

$$F \subset \bigcap_{w \in \gamma} wD,$$

then it lies in the interior of γ.

Proof. If z is on γ or in the exterior of γ, then $z \neq 0$ and there exists a point z_1 such that $z_1 \in (0, z]$, $z_1 \in \gamma$, and $[0, z_1)$ is included in the interior of γ. Hence $zz_1^{-1} \in D(\Gamma) \subset D^c$. The last fact and the hypothesis on F imply that $z \notin F$.

THEOREM. *Let* Γ *be continuous. Suppose the infinite matrix* $\|a_{nm}\|$ $(n, m = 0, 1, 2, \cdots)$ *has the properties*

(i) $\lim_{n\to\infty} \sum_{m=0}^{\infty} a_{nm} = 1$ *and*

(ii) *for a certain open and* Γ*-regular set* D, *the relation*

$$\lim_{n\to\infty} \sum_{m=0}^{\infty} a_{nm} z^{m+1} = 0$$

holds uniformly in every compact subset of D.

Then, for each power series $P(z)$ *with a positive radius of convergence, the relation*

$$\lim_{n\to\infty} \sum_{m=0}^{\infty} a_{nm} s_m(z) = P(z;\ \Gamma) \tag{1}$$

holds uniformly in each compact subset of the set

$$\Omega = \bigcap_{\substack{w \notin M \\ w \neq \infty}} wD.$$

Remark. It is easy to see that the assumptions of our theorem imply that $\Omega \subset M(P;\ \Gamma)$ (so that the right-hand side of (1) is defined), and that the set of finite points of Ω is open.

Example 1. The family Γ of all rays emanating from the point 0 is continuous and has the property that $D(\Gamma) = \{x \mid x \geq 1\}$. In this special case, $M(P;\ \Gamma)$ is the ordinary Mittag-Leffler star of $P(z)$, and our theorem is a generalization of Okada's theorem. Here we have the additional result about the uniform summability in compact subsets, which was proved for special domains D in [2] (see also [1, p. 189]).

Example 2. Let γ be a Jordan arc defined (for $z = re^{i\phi}$) by $\phi = \phi(r)$, where $\phi(r)$ is continuous for $0 \leq r < \infty$. The family of all Jordan arcs of the form $\gamma_\alpha = e^{i\alpha}\gamma$

$(0 \leq \alpha < 2\pi)$ is continuous. In the particular case where γ is the polygonal line composed of the line segment $[0, 1]$ and the ray $1 - iy$ $(0 \leq y < +\infty)$,

$$D(\Gamma)^c = \{z \mid \Re z \geq 1, \ \Im z \leq 0\}.$$

To prove the theorem, suppose that F is any compact set in Ω and that $0 \in F$. First we establish the existence of a rectifiable Jordan curve γ with the three properties

(a) $\gamma \subset M(P; \Gamma)$, (b) $\Gamma\gamma^{-1} \subset D$, (c) F lies in the interior of γ.

Since $M(P; \Gamma)$ is a Γ-star set, Lemma 1 and our hypothesis on F imply that

$$F(M^c)^{-1} \subset D \quad \text{and} \quad \delta \equiv d(F(M^c)^{-1}, D^c) > 0.$$

Because the set $(M^c)^{-1}$ is a bounded continuum, there corresponds to each $a > 0$ a rectifiable Jordan curve ξ that includes $(M^c)^{-1}$ in its interior and has the property

$$\sup_{w \in \xi} d(w, (M^c)^{-1}) < \delta/4a.$$

Let $\gamma = \xi^{-1}$. Then γ obviously has property (a).

Since F is bounded (say $|z| \leq a$ for all $z \in F$), there corresponds to each $u \in \gamma$ a point $w = w(u) \in M^c$ such that $|u^{-1} - w^{-1}| < \delta/4a$, whence $|z/u - z/w| < \delta/4$ for all $z \in F$. Thus

$$d(z/u, D^c) \geq d(z/w, D^c) - |z/u - z/w| > \delta/2.$$

Therefore $d(F\gamma^{-1}, D^c) > \delta/2$. In particular, γ has property (b).

Since property (b) is equivalent to the assumption that $F \subset \Omega$, Lemma 2 implies that γ has property (c).

Lemma 1, the properties of γ, the fact that $1 \notin D$, the assumption (ii) of our theorem, and the calculus of residues yield the relation

$$P(z; \Gamma) = \frac{1}{2\pi i} \oint_\gamma \frac{P(w; \Gamma)}{w} \lim_{n \to \infty} \sum_{m=0}^{\infty} a_{nm} \left[1 - \left(\frac{z}{w}\right)^{m+1}\right] \left(1 - \frac{z}{w}\right)^{-1} dw$$

$$= \lim_{n \to \infty} \sum_{m=0}^{\infty} a_{nm} s_m(z)$$

for all $z \in F$, and the convergence is uniform in F.

REFERENCES

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2. Y. Okada, *Über die Annäherung analytischer Functionen*, Math. Z. 23 (1925), 62-71.

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