ON THE CENTROID OF A HOMOGENEOUS WIRE

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1. INTRODUCTION

Let C be a closed convex curve in the Euclidean plane E_2. If C has continuous curvature, then the *curvature centroid* of C is defined as the center of mass of C considered as a wire whose density is equal to the curvature at each point. Hayashi [5] shows that at least four normals of C pass through its curvature centroid. Bose and Roy [2] and Tietze [6] prove that the *area centroid* of C has the same property (the area centroid is the center of mass of a disk of uniform density bounded by C). In this paper, we prove that the *perimeter centroid* of C also has this property (Section 4). (The point (x_0, y_0) is the perimeter centroid of C if

$$Lx_0 = \int_C x\,ds, \qquad Ly_0 = \int_C y\,ds, \tag{1}$$

where L is the length of C, and s is arc length along C.) The proofs in [2] and [5] employ Fourier series and put restrictions on the smoothness of C; however, even if C is assumed smooth, this technique fails to give the result for the perimeter centroid. Indeed, Bose and Roy [3] obtain by these methods only the weaker result that if m is the number of points on C where the radius of curvature is equal to three times the support function with respect to (x_0, y_0), and n is the number of normals through (x_0, y_0), then $m + n \geq 4$. The proof we give in Section 4, like that of Tietze [6] in the case of the area centroid, is purely geometric and places no smoothness restrictions on C.

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2. DEFINITIONS

A *support line* of C is a line intersecting C so that the interior of C lies entirely on one side of the line.

A line or line segment ℓ containing a point P of C is a *normal* if and only if ℓ is orthogonal to some support line of C through P.

Let C_1 be an arc lying in the upper half-plane and having its endpoints at (-a, 0) and (a, 0) on the x-axis. C_1 is a *convex arc* if and only if, together with its chord from (-a, 0) to (a, 0), it forms a closed convex curve.

The *moment* of C_1 about the x-axis, denoted by $I(C_1)$, is given by

$$I(C_1) = \int_{C_1} y\,ds. \tag{2}$$

The integralgeometric definition of the area A(S) of a surface S in Euclidean space E_3 is as follows: let dG be the usual integralgeometric density for the set of

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lines G in E_3 (see [1, p. 65] or [4, p. 88]). For each line G, let $n(G \cap S)$ be the number of points of intersection of G with S. Then

$$A(S) = \frac{1}{\pi}\int n(G \cap S)dG, \tag{3}$$

where the integration is over all lines G. It is proved in [1, p. 69], that this formula gives the "usual" surface area under suitable smoothness conditions on S. Moreover, in the case of convex surfaces the formula is consistent with the usual definition, without any smoothness restrictions (see [4, p. 95]). If S is the surface of revolution generated by revolving a convex arc C about its chord, (3) is still consistent, as can be established by approximating C with inscribed polygons and considering the corresponding surfaces of revolution; however, in the proof of Lemma 1, below, we need (3) in this last case only when C is a polygon.

3. PRELIMINARY LEMMAS

LEMMA 1. *Let* C_1 *and* C_2 *be convex arcs in the upper half-plane, each having its endpoints at* (-a, 0) *and* (a, 0). *Suppose that* C_2 *lies below* C_1 *and also in the strip* $\{(x, y): -a \leq x \leq a\}$. *Then, if* $C_1 \neq C_2$,

$$I(C_2) < I(C_1). \tag{4}$$

Proof. If S_1 and S_2 are the surfaces generated by revolving C_1 and C_2, respectively about the x-axis, then S_2 is a convex surface contained within S_1, and $S_1 \neq S_2$. Moreover, the area of S_i is given by

$$A(S_i) = 2\pi I(C_i) \qquad (i = 1, 2). \tag{5}$$

For any line G, $n(G \cap S_2) \leq n(G \cap S_1)$, and there exists a set of lines of positive measure intersecting S_1 but not intersecting S_2. Thus, the validity of formula (3) for $S = S_1$ would by (5) imply the required result (4). Instead of establishing (3) with $S = S_1$, we proceed as follows:

Let P_1 be a polygonal arc circumscribed about C_1 and having the same endpoints as C_1. Let R_1 be the surface generated by rotating P_1 about its chord. Then R_1 is the union of "conical segments" (cones and truncated cones), for each of which (3) is consistent. But the integral in (3) is additive, so that (3) is consistent for R_1. Hence we can apply the argument of the last paragraph, with C_1 replaced by P_1, to obtain the inequality

$$I(C_2) \leq I(P_1) - \delta, \tag{6}$$

where δ is positive and independent of P_1. (To establish the existence of δ, one may consider the convex surface S_2' formed by taking the convex hull of S_2 and a point outside S_2 but inside S_1. Then $A(S_2) + 2\pi\delta = A(S_2') \leq A(R_1)$.) Now the continuity of I(C), as a functional on convex arcs C, implies, by (6),

$$I(C_2) \leq I(C_1) - \delta < I(C_1). \tag{7}$$

We shall also use the following well-known, elementary lemma.

LEMMA 2. *Let* C *be a closed convex curve with the origin interior to* C, *and let* $r = r(\theta)$ $(0 \leq \theta \leq 2\pi)$ *be the polar equation of* C. *Suppose* $r(\theta)$ *has a relative maximum or minimum. Then the corresponding radius is normal to* C.

4. A FOUR-NORMAL THEOREM

We now are ready to obtain the result described in the Introduction.

THEOREM. *At least four normals to a closed convex curve pass through its perimeter centroid.*

Proof. Assume that the perimeter centroid of C lies at the origin, so that

$$\int_C y\,ds = 0, \tag{8}$$

and let $r = r(\theta)$ $(0 \leq \theta \leq 2\pi)$ be the polar equation of C. If we can show that $r(\theta)$ has at least four relative maxima and minima, the theorem is proved, by Lemma 2. By periodicity, $r(\theta)$ must have an even number of extrema, so it suffices to show it is impossible that $r(\theta)$ has exactly one relative maximum and one relative minimum.

The continuity of $r(\theta)$ implies $r(\theta_0 + \pi) = r(\theta_0)$ for some θ_0. By performing a rotation, we may suppose $\theta_0 = 0$, so that $r(0) = r(\pi) = a$. Suppose that $r(\theta)$ has exactly one maximum, assumed for $0 < \theta < \pi$, and one minimum, necessarily assumed for $\pi < \theta < 2\pi$. Then $r(\theta) > a$ if $0 < \theta < \pi$, and $r(\theta) < a$ if $\pi < \theta < 2\pi$. (Note that $r(\theta) \neq a$ for $0 < \theta < \pi$, since otherwise at least two extrema would exist in $0 < \theta < \pi$. The same holds for $\pi < \theta < 2\pi$.) It follows that the reflection across the x-axis of the part of C in the lower half-plane is a convex arc C_2 that lies below the arc C_1 of C in the upper half-plane. Moreover, C_2 lies in the disk $\{(x, y): x^2 + y^2 \leq a^2\}$, hence in the strip $\{(x, y): -a \leq x \leq a\}$. So by Lemma 1, $I(C_2) < I(C_1)$. But then

$$0 < I(C_1) - I(C_2) = \int_C y\,ds; \tag{9}$$

this contradicts (8) and completes the proof.

5. SOME REMARKS

Remark 1. The example of the ellipse shows that the number four in the theorem cannot be replaced by a larger number.

Remark 2. The converse of Lemma 2 is false, so one cannot conclude in general that the number of normals through a point is even. Indeed, we show how one can construct, for any integer $n \geq 5$, a convex curve C with exactly n normals through its perimeter centroid.

If n is even $(n \geq 6)$, a regular polygon of $n/2$ sides does the trick. For n odd, we proceed as follows. In the (x, y)-plane, let

$$Q = (0, 1), \quad R = (a, 1), \quad S = (b, 0), \quad T = (0, -1),$$

where $0 < a < b$. Let C' be the unit semicircle $\{(x, y): x^2 + y^2 = 1, x \leq 0\}$. Let C be formed by C' together with the segments QR, RS, and ST. Continuity considerations show that for the number a small, and a proper choice of b, the perimeter centroid P lies on the line $x = a$ and in the upper half-plane. Exactly five normals of C pass through P (note that there is a normal at R not corresponding to an extremum of the radial function). If one replaces C' by a rectangular arc C'' in the left half-plane with a pair of vertices at Q and T, and a side parallel to the y-axis, such that C' and C'' have the same x- and y-moments, then the perimeter centroid P is unchanged. The new curve has seven normals passing through P. It is clear that suitable variations of C' in this manner yield examples for any odd $n \geq 5$.

Remark 3. Tietze's proof of the four-normal property for the area centroid is valid for any curve that is star-shaped with respect to its area centroid. On the other hand, the following example shows that, in general, even the simplest star-shaped, nonconvex curves may fail to have four normals through the perimeter centroid.

Let K be the circle $x^2 + y^2 = 1$. Let $Q = (0, 1/2)$. Let $R = (a, \sqrt{1 - a^2})$ $(0 < a < 1)$, and let R' be the reflection of R across the y-axis. Let C be the curve consisting of the larger arc RR' of K together with the segments $R'Q$ and QR (so that C is a "notched" circle). If a is chosen small enough, the perimeter centroid P of C lies on the positive half of the y-axis, C is star-shaped with respect to P, and exactly *two* normals of C pass through P. By slightly rounding off the three corners of C, one obtains an infinitely differentiable example.

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