THE MEASURABILITY OF AN INTRINSIC LENGTH

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INTRODUCTION

Let X be a compact Hausdorff space. Let $f: X \to E_n$ be a continuous mapping. Let τ be a real-valued continuous function defined on $f(X) \subset E_n$. As is well known, the function $\phi(x) = \tau[f(x)]$ $(x \in X)$ induces an upper-semicontinuous decomposition of X into closed disjoint "level sets" or *contours*. The *lower contour* $C_\phi^-(t)$, respectively, *upper contour* $C_\phi^+(t)$, is defined as the boundary in X of the open set $D_\phi^-(t)$, respectively, $D_\phi^+(t)$. While the "level sets" may contain interior points, the lower and upper contours are closed, nowhere dense subsets of X.

If X is a finitely triangulable m-dimensional space and therefore a space on which Lebesgue m-area $L_m(f)$ for mappings into E_n is defined, neither the contours nor the upper or lower contours need inherit the latter property of the space X. Even in the case where $m = 2$ and X is a closed, simply-connected Jordan region J in the plane, the concept of Lebesgue 1-area, or length, is not in general available for the partial mappings $f \mid C_\phi^-(t)$, although $L_2(f)$ is defined.

It is partly because of the general nature of these upper and lower contours as sets and partly because of the unavailability of appropriate definitions of k-area sufficiently general for application to mappings from arbitrary compact spaces, that in each of the several versions of the Cavalieri inequality

$$\int_{-\infty}^{\infty} A_{m-1}(f, C_\phi^-(t))dt \leq KL_m(f)$$

that have been proved [1, 2, 3] the corresponding notion of $(m - 1)$-area $A_{m-1}(f, C_\phi^-(t))$ depends on concepts involving $D_\phi^-(t)$, not only $C_\phi^-(t)$. In this sense, these $(m - 1)$-areas are not intrinsic definitions, nor are they applications of a general concept of k-area.

Definitions of Lebesgue k-area of scope sufficient to apply to mappings from the lower (or upper) contours have recently been introduced by R. F. Williams [6]. With a view towards possible use of these as intrinsic $(m - 1)$-areas in a Cavalieri inequality, we concern ourselves in this paper with the case $m = 2$ and prove that the function $L_1^P(f \mid C_\phi^-(t))$, defined in terms of Williams's 1-area L_1^P, has the basic property of being measurable as a function of t $(-\infty < t < \infty)$.

1. NOTATION AND DEFINITIONS

An *open cover* for a topological space X is a collection of open subsets of X whose union is X. A finite open cover α is said to be of *order at most* m if no point of X is covered by more than m distinct elements of α. A compact

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Hausdorff space X is said to be of *dimension at most* m ($\dim X \leq m$) if every open cover U has a finite refinement of order at most $m + 1$.

If X is a topological space and α is a finite open cover for X, the *nerve* X_α of α is any finite simplicial complex K whose vertices are in one-to-one correspondence with the elements of α and which is such that a set $(v_0, v_1, \cdots, v_k)$ of its vertices is the set of vertices of a simplex of K if and only if the corresponding elements $A_0, A_1, \cdots, A_k$ of α have a nonempty intersection.

We regard the nerves of open covers and in fact all complexes as geometric complexes situated in some Euclidean space. Simplexes are always open simplexes unless the word "simplex" is qualified. The *body* (or underlying space) $|K|$ of a geometric complex K is the union of all the simplexes of K.

In connection with an open cover α of X and a given geometric realization of its nerve X_α, we shall use the following notation. Let V_α denote the collection of all vertices of X_α. The elements of α are indexed in a one-to-one fashion by the elements of V_α under the correspondence defining the vertices of the nerve. Thus, we may write $\alpha = \{A_v \mid v \in V_\alpha\}$. For each $x \in X$, let $V(x)$ be the collection of all vertices $v \in V_\alpha$ such that $x \in A_v$. Let $(V(x))$ and $[V(x)]$ denote, respectively, the open and closed simplexes of X_α spanned by the elements of $V(x)$.

By "mapping" or "map" we shall always mean a "continuous mapping." A *canonical map* $g: X \to |X_\alpha|$ is any mapping with the property that $g(x) \in [V(x)]$ for each $x \in X$. A *star-canonical map* $g: X \to |X_\alpha|$ is any map with the property that $g(x) \in (V(x))$ for each $x \in X$. (Star-canonical mappings are identical with the barycentric α-maps of [5].)

If X is a compact Hausdorff space, $\dim X \leq m$, and E_n is Euclidean n-space, then a triple (α, g, h) is said to be an m-*canonical map triple* (m-*star canonical map triple*) of X into E_n, provided

(a) α is a finite open cover for X of order at most $m + 1$,

(b) $g: X \to |X_\alpha|$ is a canonical map (star-canonical map) and

(c) $h: X_\alpha \to E_n$ is simplicial (relative to a triangulation of E_n). We write $(\alpha, g, h): X \to E_n$.

If m is a positive integer, K and L are geometric complexes, and $h: K \to L$ is a simplicial map, then the *elementary* m-*area* $e_m(h)$ is defined to be the number

$$e_m(h) = \sum a_m[h(\sigma)],$$

where the sum is taken over all m-simplexes σ of K, and where $a_m[h(\sigma)]$ denotes the elementary m-area (computable by determinants) of the simplex $h(\sigma)$ ($a_m[h(\sigma)] = 0$ if $h(\sigma)$ has combinatorial dimension less than m).

If X is a compact *metric space* of dimension at most m and $f: X \to E_n$ is a continuous mapping, then R. F. Williams [6] defines $L_m^p(f)$ to be the smallest number r with the property that for every $\varepsilon > 0$, there exists an m-canonical map triple $(\alpha, g, h): X \to E_n$ such that mesh $\alpha < \varepsilon$, $\|f - hg\| < \varepsilon$ and $e_m(h) < r + \varepsilon$. Here mesh $\alpha = \sup\{\text{diameter } A \mid A \in \alpha\}$, and $\| \ \|$ is the usual norm defining the uniform topology for the set of all mappings of X into E_n.

It will be convenient to work with the following slight modification of Williams's original definition.

$L_m^p(f)$ is the smallest number r with the property that for every $\varepsilon > 0$, there exists an m-star canonical map triple (α, g, h): $X \to E_n$ such that mesh $\alpha < \varepsilon$, $\|f - hg\| < \varepsilon$, and $e_m(h) < r + \varepsilon$.

That this definition is equivalent to the foregoing is not difficult to prove. The key idea is that the star-canonical maps are uniformly dense in the space of all canonical maps of X into $|X_\alpha|$. This may be established by arguments based on those given in [5, p. 72, footnote †].

2. CONVERGENCE PROPERTIES OF LOWER CONTOURS AND EXTENSION OF COVERS

Let X be a compact, connected, locally connected Hausdorff space. Let ϕ be a nonconstant, real-valued, continuous function on X. For each t $(-\infty < t < \infty)$, let

$$C_\phi(t) = \{x \in X \mid \phi(x) = t\},$$

$$D_\phi(t) = \{x \in X \mid \phi(x) < t\},$$

$$C_\phi^-(t) = \text{Bdry } D_\phi(t).$$

In case X is a subset of a larger space, topological operations on subsets of X, such as closure and boundary, are understood to be relative to the subspace topology of X.

Since X is a continuum, $\phi(X) = \{t \mid t_{min} \leq t \leq t_{max}\}$, where t_{min} and t_{max} are the smallest and largest values assumed by ϕ on X. None of the sets $C_\phi^-(t)$ is empty for t in the range $t_{min} < t < t_{max}$. Unless we state the contrary, we always assume that t lies in this range.

LEMMA 1. *If* $x \in C_\phi^-(t)$ *and* U_x *is any open set containing* x, *then there exists a* $\bar{t}$ *such that for each* t' $(\bar{t} < t' < t)$ *the relation* $C_\phi^-(t') \cap U_x \neq \emptyset$ *holds.*

Proof. Let V_x be a connected open set containing x and such that $\overline{V}_x \subset U_x$. There exists a $y \in V_x$ such that $f(y) < t$. Let $\bar{t} = f(y)$, and let $\bar{t} < t' < t$. Since the function $\phi \mid V_x$ assumes the values $\bar{t}$ and t, $C_\phi(t') \cap V_x \neq \emptyset$. Let $\{t_n\}$ be a sequence of real numbers such that $t_n < t_{n+1}$, $\bar{t} < t_n < t'$, $t_n \to t'$ as $n \to \infty$. For each n, choose a point $x_n \in C_\phi(t_n) \cap V_x$. The sequence $\{x_n\}$ is an infinite set contained in $D_\phi(t') \cap V_x$, and it has a limit point x_0 in $\overline{D_\phi(t')} \cap \overline{V}_x$ with the property $f(x_0) = t'$. Hence $x_0 \in \overline{D_\phi(t')} - D_\phi(t') = C_\phi^-(t')$, and therefore $C_\phi^-(t') \cap U_x \neq \emptyset$.

LEMMA 2. *If* U *is an open set such that* $C_\phi^-(t) \subset U$, *then there exists a* $\bar{t}$ *such that* $C_\phi^-(t') \subset U$ *for each* t' $(\bar{t} < t' < t)$.

Proof. Let $\psi = \phi \mid \overline{D_\phi(t)}$, and for each t' $(t_{min} \leq t' \leq t)$ let

$$C_\psi(t') = \overline{D_\phi(t)} \cap C_\phi(t').$$

The collection $\{C(t')\}$ $(t_{min} \leq t' \leq t)$ is an upper-semicontinuous decomposition of $\overline{D_\phi(t)}$ induced by ψ. Since $C_\psi(t) = C_\phi^-(t)$ and $C_\phi^-(t) \subset U \cap \overline{D_\phi(t)}$, there exists (by the characteristic property of upper-semicontinuous decompositions) a set W, open in X, such that

$$C_\psi(t) \subset W \cap \overline{D_\phi(t)} \subset U \cap \overline{D_\phi(t)},$$

and with the property that if $C_\psi(t') \cap W \cap \overline{D_\phi(t)} \neq \emptyset$, then $C_\psi(t') \subset U \cap \overline{D_\phi(t)}$. Let $x \in C_\phi^-(t)$. By Lemma 1, there exists a $\bar{t}$ such that if $\bar{t} < t' < t$, then $C_\phi^-(t') \cap W \neq \emptyset$. Since $C_\phi^-(t') \subset C_\phi(t') = C_\psi(t')$, it follows that $C_\psi(t') \cap W \cap \overline{D_\phi(t)} \neq \emptyset$. Consequently $C_\phi^-(t') \subset U$.

LEMMA 3. *Let* X *be a compact, connected, locally connected metric space. Let* $f\colon X \to E_n$. *Let* ϕ *be any nonconstant, real-valued, continuous function on* X, *and let* $f_s = f\,|\,C_\phi^-(s)$. *Suppose* t *is some fixed number for which* $0 \leq \dim C_\phi^-(t) \leq 1$. *Then to every* $\varepsilon > 0$ *there corresponds a* $\delta > 0$ *such that if* $(\alpha, g, h)\colon C_\phi^-(t) \to E_n$ *is a* 1-*star canonical map triple for which* $\|f_t - hg\| < \delta$ *and* mesh $\alpha < \delta$, *then there exists a* t^* *such that for each* t' $(t^* < t' \leq t)$ *there exists a* 1-*star canonical map triple* $(\alpha', g', h')\colon C_\phi^-(t') \to E_n$ *such that* $\|f_{t'} - h'g'\| < \varepsilon$, mesh $\alpha' < \varepsilon$, *and* $e_1(h') \leq e_1(h)$.

Proof. Let $\varepsilon > 0$ be given. Let δ_0 be the modulus of continuity of f on X corresponding to $\varepsilon/3$. Let $\delta = \min(\delta_0, \varepsilon/5)$. For brevity, set $C = C_\phi^-(t)$, and let $(\alpha, g, h)\colon C \to E_n$ be a 1-star canonical map triple for which $\|f_t - hg\| < \delta$ and mesh $\alpha < \delta$. We may assume that the elements A of α, each open in C, are indexed in a one-to-one fashion by the vertices v of C_α, and we write $\alpha = \{A_v\} = \{A_v \mid v \in V_\alpha\}$.

Since we regard C_α as a geometric complex in E_k, the point set $|C_\alpha|$ is a polyhedron. By two theorems of K. Borsuk [4, p. 30], $|C_\alpha|$ is an absolute neighborhood retract and there exists an open set $U \supset C$ and a mapping $\bar{g}\colon U \to |C_\alpha|$ such that $\bar{g}\,|\,C = g$. Since X is a normal space, we could assume that U is such that $\bar{g}$ is also defined on $\overline{U}$. However, for our purposes we need only to regard $\bar{g}$ as defined on U and uniformly continuous on U.

For each vertex $v \in C_\alpha$, let $U_v = \bar{g}^{-1}[\mathrm{St}(v)]$, where $\mathrm{St}(v)$ is the open star at v. Since $\{\mathrm{St}(v)\}$ is a finite open cover for $|C_\alpha|$ of order at most 2, and since g is star-canonical with respect to α, the family $\{U_v\}$ is a finite open cover for U, of order at most 2, none of whose members is empty.

We first prove three assertions.

(a) $$A_v = U_v \cap C = g^{-1}[\mathrm{St}(v)].$$

The equality $C \cap U_v = C \cap \bar{g}^{-1}[\mathrm{St}(v)] = g^{-1}[\mathrm{St}(v)]$ is immediate. The equality $A_v = g^{-1}[\mathrm{St}(v)]$ is a characteristic property of star-canonical mappings.

(b) *If* v *and* w *are distinct vertices of* C_α, *then* $U_v \cap U_w \neq \emptyset$ *if and only if* $A_v \cap A_w \neq \emptyset$.

It is clear that if v and w are distinct vertices of C_α and $A_v \cap A_w \neq \emptyset$, then $U_v \cap U_w \neq \emptyset$. On the other hand, if $U_v \cap U_w \neq \emptyset$, let $z \in U_v \cap U_w$. It follows that $\bar{g}(z) \in \mathrm{St}(v) \cap \mathrm{St}(w) = (v, w) \in C_\alpha$ and therefore $A_v \cap A_w \neq \emptyset$.

(c) $\bar{g}\colon U \to |C_\alpha|$ *is star-canonical with respect to* $\{U_v\}$.

If v and w are distinct and $z \in U_v \cap U_w$, then $\bar{g}(z) \in (v, w) \in C_\alpha$. If $z \in U_v$ for one and only one vertex $v \in C_\alpha$, then $\bar{g}(z) \in \mathrm{St}(v)$ but $\bar{g}(z)$ is in no 1-simplex of C_α of the form (v, w). Since $\mathrm{St}(v)$ is the disjoint union of v and the 1-simplexes of C_α (if any) of the form (v, w), then $\bar{g}(z) = v$.

Now let d be a metric for X. Let $m = \min(\delta, \delta_1)$, where δ_1 is the modulus of continuity of $h\bar{g}$ on U. For each $x \in C$, let r_x be a real number such that $0 < r_x < m$ and such that the spherical neighborhood about x,

$$S(x, r_x) = \{x' \in X \mid d(x', x) < r_x\},$$

has the property that if $x \in A_v$, then $S(x, r_x) \subset U_v$. We remark that this implies that if v and w are distinct and $x \in A_v \cap A_w$, then $S(x, r_x) \subset U_v \cap U_w$.

We now define a "thinner" open set containing C as well as a related open cover and a mapping, and we prove five propositions concerning these. Let

$$G = \bigcup\{S(x, r_x) \mid x \in C\}, \quad G_v = G \cap U_v, \quad g^* = \bar{g} \mid G.$$

We make two observations.

(1) $\gamma = \{G_v\} = \{G_v \mid v \in V_\alpha\}$ is an open cover for G of order at most 2.

(2) $$A_v = A_v \cap G = G \cap U_v \cap C = G_v \cap C.$$

The first is clear, and the second follows from (a) and the definition of G. We also see that $G_v \neq G_w$ if v and w are distinct.

(3) If v and w are distinct, then $G_v \cap G_w \neq \emptyset$ if and only if $A_v \cap A_w \neq \emptyset$.

This is a consequence of (b) and the fact that $G_v \cap G_w \neq \emptyset$ implies that $U_v \cap U_w \neq \emptyset$, as well as of (2).

(4) $g^*: G \to |C_\alpha|$ is star-canonical with respect to $\gamma = \{G_v\}$.

Suppose v and w are distinct and that $z \in G_v \cap G_w$. Then $z \in U_v \cap U_w$, and it follows that $g^*(z) \in (v, w) \in C_\alpha$. If we suppose that $z \in G_v$ and that, for each vertex u distinct from v, $z \notin G_u$, then since $G_v = G \cap U_v$ it is certainly true that $z \in U_v$. Now suppose that w is distinct from v and that $z \in U_w$. Then since $z \in G$, it follows that $z \in G_w$, and w and v can not be distinct. Hence z cannot lie in any element of the collection $\{U_v\}$ other than U_v. It follows that $z \in \bar{g}^{-1}(v)$. It is now clear that $g^*(z) = v$.

(5) $$\operatorname{mesh} \gamma < 5m.$$

If y and y' are any two points in G_v, then there exist vertices u and w of C_α and points $x \in A_u$ and $x' \in A_w$ such that $y \in S(x, r_x) \subset G_u$ and $y' \in S(x', r_{x'}) \subset G_w$. Since $G_u \cap G_v \neq \emptyset$ and since $G_v \cap G_w \neq \emptyset$, it follows that $A_u \cap A_v \neq \emptyset$ and $A_v \cap A_w \neq \emptyset$. Let $x'' \in A_u \cap A_v$ and $x''' \in A_v \cap A_w$. Since

$$d(y, y') \leq d(y, x) + d(x, x'') + d(x'', x''') + d(x''', x') + d(x', y'),$$

it follows that $d(y, y') < 5m$.

The collection γ will induce an open cover on a lower contour sufficiently close to C. Since $C \subset G$, there exists by Lemma 2 a $\bar{t}$ such that for each t'' $(\bar{t} < t'' < t)$ we are assured that $C_\phi^-(t'') \subset G$. Furthermore, for each pair of distinct vertices $u, v \in V_\alpha$ for which $G_u \cap G_v \neq \emptyset$, there exists a number $t_{u,v}$ such that if t' satisfies the inequality $\bar{t} \leq t_{u,v} < t' < t$, then $C_\phi^-(t') \cap G_u \cap G_v \neq \emptyset$. This is guaranteed by Lemma 1 and the property (3). If t^* is the largest of the numbers $t_{u,v}$ among all pairs of vertices for which $G_u \cap G_v \neq \emptyset$, $u \neq v$, then $\bar{t} \leq t^* < t$, and if t' is such that $t^* < t' < t$, then for each pair of distinct vertices $u, v \in V_\alpha$ for which $G_u \cap G_v \neq \emptyset$, we are assured that $C_\phi^-(t')$ meets $G_u \cap G_v$.

Suppose that t' is some fixed number satisfying the inequality $t^* < t' < t$. We set $C' = C_{\phi}^{-}(t')$ for brevity, and we define an open cover for C' in the following way. For each vertex $v \in C_\alpha$, let $A'_v = G_v \cap C'$ and let $\alpha' = \{A'_v\}$. The cover α' has the property

(6) if u and v are distinct vertices of C_α, then $A_u \cap A_v \neq \emptyset$ if and only if $A'_u \cap A'_v \neq \emptyset$.

The collection $\alpha' = \{A'_v\}$ is not in general in one-to-one correspondence with the collection $\alpha = \{A_v\}$, since there may exist distinct vertices v and w for which $A'_v = A'_w$. Should this happen for some pair of vertices v and w, then A'_v meets no element of α' distinct from itself. To show this, we observe that if v and w are distinct and $A'_v = A'_w$, then $A'_v = G_v \cap G_w \cap C' = A'_w$. If $A'_v \cap A'_u \neq \emptyset$, then

$$A'_v \cap A'_u = G_v \cap G_w \cap G_u \cap C' \neq \emptyset .$$

Since $\{G_v\}$ has order at most 2, this is impossible if u, v, and w are all distinct. Hence $u = v$, or $u = w$. This argument also shows that for a given vertex v, there is at most one vertex u distinct from v for which $A'_v = A'_u$. In what follows, we shall not regard $A'_v = A'_w$ as distinct elements of the cover α', should such a phenomenon occur.

We construct a realization of the nerve $C'_{\alpha'}$ of α' in the body of the complex C_α in the following way. For each $A'_v \in \alpha'$, let the vertex associated with A'_v be v, unless there is a vertex w, distinct from v (necessarily at most one), such that $A'_w = A'_v$. In this case, since $A_w \cap A_v \neq \emptyset$, let $p(v, w) = g(y_0)$ for some $y_0 \in A_w \cap A_v$. Since g is star-canonical, $p(v, w)$ is a point in the 1-simplex $(v, w) \in C_\alpha$. We consider it a (new) vertex and associate it with A'_v. The vertices v and w together with the 1-simplex they span are discarded; we retain only the point $p(v, w)$. To define $C'_{\alpha'}$ completely, we must specify the vertices and declare which pairs of vertices are endpoints of 1-simplexes. The vertices of $C'_{\alpha'}$ are the vertices v and the points $p(v, w)$, selected as above, that remain after possible discards. The 1-simplexes of $C'_{\alpha'}$ are the 1-simplexes of C_α that remain (if any). Accordingly, a pair of vertices of $C'_{\alpha'}$ are the vertices of a 1-simplex of $C'_{\alpha'}$ if they are the vertices of a 1-simplex of C_α that has not been discarded.

The process above will be completely defined when we have shown that no discarded vertex of C_α can be the vertex of a 1-simplex of C_α whose other vertex is not also discarded. Indeed, if u and v are distinct vertices of C_α such that $A_u \cap A_v \neq \emptyset$, then $G_u \cap G_v \neq \emptyset$. Since $C' \cap (G_u \cap G_v) \neq \emptyset$, it follows that $(C' \cap G_u) \cap (C' \cap G_v) \neq \emptyset$; in other words, $A'_u \cap A'_v \neq \emptyset$. If v has been discarded, then v must be the endpoint of a 1-simplex $(v, w) \in C_\alpha$ which has been discarded along with the endpoint w. Hence $A'_v = A'_w$. Since there is at most one vertex u distinct from v for which $A'_v = A'_u$, it follows that $u = w$. This argument also shows that A_v and A_w must be an isolated pair in the cover α for C, in the sense that A_v and A_w meet but neither meets an element of α distinct from both. There is no 1-simplex in C_α with either of the vertices v or w except (v, w). Our construction may be viewed as the replacement of certain isolated 1-simplexes by points.

We define a map $g': C' \to |C'_{\alpha'}|$ and show that g' is star-canonical with respect to α'. Let $x \in C'$. If $x \in A'_v$ and there is no vertex w distinct from v such that $A'_v = A'_w$, let $g'(x) = g^*(x)$. If there is such a vertex w, let $g'(x) = p(v, w)$. That g is star-canonical with respect to α' is a consequence of the fact that g^* is star-canonical with respect to $\{G_v\}$ and of the fact that $A'_v = A'_w$ is regarded as a single element of α'. That g' is continuous is a consequence of the continuity of g^* and of the fact that whenever $A'_v = A'_w$, then A'_v is isolated from the other elements of α'.

We now define the mapping $h'\colon C'_{\alpha'} \to E_n$ by the equation $h' = h \mid |C'_{\alpha'}|$. Since $|C'_{\alpha'}| \subset |C_\alpha|$, the equation has meaning, and clearly h' is simplicial (relative to a subdivision of $h(|C_\alpha|)$). Since $g^*(C') \subset |C_\alpha|$ and $g'(C') \subset |C'_{\alpha'}|$, the composition $h' g'$ also has meaning.

Next we show that $\|f_{t'} - h' g'\| < \varepsilon$ and that mesh $\alpha' < \varepsilon$. Suppose that $x \in A'_v$ and that there is no vertex w distinct from v for which $A'_v = A'_w$. Since $g'(x) = g^*(x)$ and $x \in G_v$, there exists a $y \in C$ such that $x \in S(y, r_y) \subset G \subset U$. Now

$$|h'g'(x) - f(x)| = |hg^*(x) - f(x)| \leq |hg^*(x) - hg^*(y)| + |hg^*(y) - f(y)| + |f(y) - f(x)|,$$

and since $d(x, y) < r_y < m \leq \min(\delta, \delta_1)$, it follows that

$$|f(y) - f(x)| < \frac{\varepsilon}{3}, \qquad |hg^*(x) - hg^*(y)| < \frac{\varepsilon}{3}, \quad \text{and } |hg^*(y) - f(y)| < \frac{\varepsilon}{5}.$$

If there is a vertex w distinct from v for which $A'_v = A'_w$, then

$$g'(x) = p(v, w) = g^*(y_0),$$

where y_0 is our preselected point in $A_v \cap A_w$. Since $x \in G_v \cap G_w$, there exists a $y \in C$ such that $x \in S(y, r_y)$ *and* $y \in A_v \cup A_w$. Indeed, since $S(y, r_y) \subset U_u$ for some vertex $u \in C_\alpha$ and since $y \in C \cap U_u$, it follows that $y \in A_u$. Since $x \in S(y, r_y)$, it follows that $x \in G_u \cap C' = A'_u$. Hence $A'_u \cap A'_v \neq \emptyset$, and therefore $u = v$ or $u = w$. In either case, $y \in A_v \cup A_w$. For definiteness, suppose that $y \in A_w$. Then y and y_0 are both in A_w and $d(y, y_0) <$ mesh $\alpha < \delta$, which implies that $|f(y) - f(y_0)| < \varepsilon/3$. Now

$$|h'g'(x) - f(x)| = |hg(y_0) - f(x)| \leq |hg(y_0) + f(y_0)| + |f(y_0) - f(y)| + |f(y) - f(x)|.$$

Since $|hg(y_0) - f(y_0)| < \delta \leq \varepsilon/3$, and since $d(y, x) < r_y < m \leq \delta_0$ implies that $|f(y) - f(x)| < \varepsilon/3$, we conclude (on combining these estimates) that

$$|h' g'(x) - f(x)| < \varepsilon\,.$$

It is evident that mesh $\alpha' \leq$ mesh $\{G_v\} < \varepsilon$.

Finally, if σ' is a 1-simplex of $C'_{\alpha'}$, then σ' is also a 1-simplex σ of C_α and $h(\sigma') = h(\sigma)$. Consequently, each nonzero term in $e_1(h') = \Sigma\, a_1 [h'(\sigma')]$, where the summation is taken over all 1-simplexes of $C'_{\alpha'}$, is equal to some nonzero term in $e_1(h) = \Sigma\, a_1 [h(\sigma)]$, where the summation is over all 1-simplexes of C_α. It follows that $e_1(h') \leq e_1(h)$.

3. MEASURABILITY OF $L_1^P(f \mid C_\phi^-(t))$

Let X be a compact metric space, and let C be a compact subspace of X $(-1 \leq \dim C \leq 1)$. Let $f\colon X \to E_n$. If $\dim C = -1$, that is, if $C = \emptyset$, define $L_1^P(f \mid C) = 0$. If $\dim C \geq 0$, let

$$L_1^p(f \mid C) = \inf\{r \mid \text{for every } \varepsilon > 0, \text{ there exists a 1-star canonical map triple } (\alpha, g, h): C \to E_n \text{ such that } \|hg - f\| < \varepsilon, \text{ mesh } \alpha < \varepsilon, \text{ and } e_1(h) < r + \varepsilon\}.$$

THEOREM. *Let* X *be a compact, connected, locally connected metric space, and* ϕ *a nonconstant, real-valued continuous function on* X. *Suppose that* $0 \leq \dim C_\phi^-(t) \leq 1$ *for every* t *in the interval* (t_{min}, t_{max}). *Then for any continuous mapping* $f: X \to E_n$, $L_1^p(f \mid C_\phi^-(t))$ *is a measurable function of* t $(-\infty < t < \infty)$.

Proof. For each $t \in I = (t_{min}, t_{max})$ and each $\varepsilon > 0$, let $H[f_t](\varepsilon)$ denote the collection of all 1-star-canonical map triples $(\alpha_t, g_t, h_t): C_\phi^-(t) \to E_n$ for which $\|h_t g_t - f_t\| < \varepsilon$ and mesh $\alpha_t < \varepsilon$. Let

$$e(t, \varepsilon) = \inf\{e_1(h_t) \mid (\alpha_t, g_t, h_t) \in H[f_t](\varepsilon)\}.$$

It is not difficult to show that for each $t \in I$,

$$L_1^p(f \mid C_\phi^-(t)) = \lim_{\varepsilon \to 0} e(t, \varepsilon), \tag{7}$$

and we omit the proof.

We first show that for each $t \in I$ and each $\varepsilon > 0$, there exists a δ $(0 < \delta \leq \varepsilon)$ such that if $\delta' \leq \delta$, then $e(t', \varepsilon) \leq e(t, \delta) + \delta$ for each $t' \in [t - \delta', t]$. Indeed, given $t \in I$ and $\varepsilon > 0$, then, by Lemma 3, let $\delta_0 > 0$ be such that if $(\alpha_t, g_t, h_t) \in H[f_t](\delta_0)$, then there exists a t^* such that for each t' $(t^* < t' \leq t)$ there is a triple $(\alpha_{t'}, g_{t'}, h_{t'}) \in H[f_{t'}](\varepsilon)$ for which $e_1(h_{t'}) \leq e_1(h_t)$. Let $\delta = \min(t - t^*, \delta_0, \varepsilon)$, and suppose that $\delta' \leq \delta$ and $t' \in [t - \delta', t]$. By definition of $e(t, \delta)$, there exists a triple $(\alpha_t, g_t, h_t) \in H[f_t](\delta)$ such that $e_1(h_t) < e(t, \delta) + \delta$. Since $H[f_t](\delta) \subset H[f_t](\delta_0)$, there exists a triple $(\alpha_{t'}, g_{t'}, h_{t'}) \in H[f_{t'}](\varepsilon)$ such that $e_1(h_{t'}) \leq e_1(h_t)$. Hence $e(t', \varepsilon) \leq e(t, \delta) + \delta$.

Let

$$\bar{e}(t, \varepsilon) = \inf_{\delta' \leq \delta} \sup\{e(t', \varepsilon) \mid t' \in [t - \delta', t]\}.$$

We shall show, for each $t \in I$ and each $\varepsilon > 0$, that

$$e(t, \varepsilon) \leq \bar{e}(t, \varepsilon) \leq L_1^p(f \mid C_\phi^-(t)) + \varepsilon. \tag{8}$$

Since $e(t', \varepsilon) \leq e(t, \delta) + \delta$ for all $\delta' \leq \delta$ and all $t' \in [t - \delta', t]$, the inequality

$$\sup\{e(t', \varepsilon) \mid t' \in [t - \delta', t]\} \leq e(t, \delta) + \delta,$$

holds, and therefore

$$\bar{e}(t, \varepsilon) = \inf_{\delta' \leq \delta} \sup\{e(t', \varepsilon) \mid t' \in [t - \delta', t]\} \leq e(t, \delta) + \delta \leq L_1^p(f \mid C_\phi^-(t)) + \varepsilon.$$

Since

$$e(t, \varepsilon) \leq \sup\{e(t', \varepsilon) \mid t' \in [t - \delta', t]\}$$

for each $\delta' \leq \delta$, it follows that $e(t, \varepsilon) \leq \bar{e}(t, \varepsilon)$.

We now show that for each fixed $\varepsilon > 0$, the function $\bar{e}(t, \varepsilon)$ is upper-semicontinuous from the left, that is,

$$\limsup_{t' \to t-0} \bar{e}(t', \varepsilon) \leq \bar{e}(t, \varepsilon).$$

Let $E_r = \{t \mid \bar{e}(t, \varepsilon) < r\}$. Suppose $t \in E_r$. There exists a $\delta' \leq \delta$ such that

$$\sup\{e(t', \varepsilon) \mid t' \in [t - \delta', t]\} < r.$$

Let $\bar{t}$ satisfy the inequality $t - \delta' < \bar{t} \leq t$, and let δ_0'' be such that

$$[\bar{t} - \delta_0'', \bar{t}] \subset [t - \delta', t].$$

Then, for all $\delta'' \leq \delta_0''$,

$$\sup\{e(t'', \varepsilon) \mid t'' \in [\bar{t} - \delta'', \bar{t}]\} < r.$$

Consequently,

$$\inf_{\delta'' \leq \delta_0''} \sup\{e(t'', \varepsilon) \mid t'' \in [\bar{t} - \delta'', \bar{t}]\} < r.$$

Thus, for each r, we have shown that if $t \in E_r$, then there exists a $\delta' \leq \delta$ such that if $t - \delta' < \bar{t} \leq t$, then $\bar{t} \in E_r$.

To complete the proof of the theorem, we observe that by a theorem proved in [2, p. 325] a function upper-semicontinuous from the left is measurable. Since $\bar{e}(t, \varepsilon)$ is measurable for each $\varepsilon > 0$, we conclude from (7) and (8) that $L_1^p(f \mid C_{\bar{\phi}}(t))$ is measurable on I.

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