SOME INVARIANTS OF p-GROUPS

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1. INTRODUCTION

The purpose of this paper is to define and study a certain system of invariants of primary abelian groups without elements of infinite height. The invariants take the form of ideals in the Boolean algebra $P(\omega)$ of all subsets of the set ω of finite ordinals. It is natural to consider the existence and uniqueness of p-groups with a given associated invariant. The main results of the paper are concerned with the existence problems.

All of the groups considered in this paper are assumed to be p-primary abelian groups, where p is some fixed prime. Most of the notation is taken from [3] and [6]. If $x \in G$, the height $h_G(x)$ is defined to be the maximum k such that $x \in p^k G$ if this maximum exists, and $h_G(x) = \infty$ if $x \in p^k G$ for all k. The subgroup of all $x \in G$ with $h_G(x) = \infty$ is denoted by G^1. A subgroup H of G is *pure in* G if $h_H(x) = h_G(x)$ for all $x \in H$. We shall denote by $f_G(k)$ the kth Ulm invariant of G:

$$f_G(k) = \dim (p^k G \cap G[p]/ p^{k+1} G \cap G[p]).$$

It is convenient to adjoin the definition $f_G(\infty) = \dim (G^1 \cap G[p])$.

We consider cardinal and ordinal numbers in the sense of von Neumann; that is, an ordinal number is a set, namely, the set of all smaller ordinals. Cardinal numbers are ordinal numbers that are not equivalent to any smaller ordinal. The cardinal number of the set X is denoted by $|X|$. The set of all subsets (the power-set) of X is represented by P(X). The symbol ω denotes the first infinite ordinal, that is, the set of all finite ordinals. The letter $\mathfrak{c}$ represents the cardinal number of the continuum. The symbols $\subset$ and $\supset$ denote inclusion in the wide sense. Finally, it is convenient to write ω^+ to denote the set $\omega \cup \{\infty\}$.

2. THE INVARIANTS

We first define a general class of invariants, then focus our attention on one of particular interest.

2.1 *Definition.* $I(G) = \{k \in \omega^+ \mid f_G(k) \neq 0\}$.

Evidently, $f_G(k) \neq 0$ if and only if $p^k G \cap G[p]/ p^{k+1} G \cap G[p] \neq 0$, that is, there is an $x \in G[p]$ such that $h_G(x) = k$. Moreover, $f_G(\infty) \neq 0$ if and only if there is a nonzero $x \in G[p]$ with $h_G(x) = \infty$. Hence:

2.2 LEMMA. $I(G) = \{h_G(x) \mid x \in G[p],\ x \neq 0\}$.

2.3 COROLLARY. *If* H *is a pure subgroup of* G, *then* $I(H) \subset I(G)$.

2.4 COROLLARY. *If* H *and* K *are pure subgroups of* G *such that* $I(H) \cap I(K) = \emptyset$, *then* $H \cap K = 0$.

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Proof. Suppose that $0 \neq x \in H[p] \cap K[p]$. Then $h_H(x) = h_G(x) = h_K(x)$ belongs to both I(H) and I(K), which contradicts $I(H) \cap I(K) = 0$. Thus, $H \cap K = \emptyset$.

2.5. LEMMA. *If* H *and* K *are pure subgroups of* G *such that* $I(H) \cap I(K) = \emptyset$, *then* H + K *is pure in* G, *the sum* H + K *is direct, and* $I(H + K) = I(H) \cup I(K)$.

Proof. Let $0 \neq z \in (H + K)[p]$. Then $z = x + y$ $(x \in H,\ y \in K)$. To prove that H + K is pure, it suffices to show that $h_{H+K}(z) \geq h_G(z)$. If $x = 0$ or $y = 0$, this is clear because of the purity of H and K. Assume that $x \neq 0$ and $y \neq 0$. Then $p(x + y) = 0$ implies that $px = -py \in H \cap K = 0$ by Corollary 2.4. By Lemma 2.2,

$$h_G(x) = h_H(x) \in I(H) \qquad \text{and} \qquad h_G(y) = h_K(y) \in I(K)\,.$$

Since $I(H) \cap I(K) = \emptyset$, it follows that $h_G(x) \neq h_G(y)$. Therefore,

$$\begin{aligned} h_G(z) = h_G(x + y) &= \min\{h_G(x), h_G(y)\} = \min\{h_H(x), h_K(y)\} \\ &\leq \min\{h_{H+K}(x), h_{H+K}(y)\} \\ &\leq h_{H+K}(x + y) = h_{H+K}(z)\,. \end{aligned}$$

Consequently, H + K is pure in G. Since $H \cap K = 0$ by Corollary 2.4, it follows that $H + K = H \oplus K$, and $f_{H+K}(k) = f_H(k) + f_K(k)$. Therefore, $I(H + K) = I(H) \cup I(K)$.

If B is a basic subgroup of G, then $f_B(n) = f_G(n)$ for all nonnegative integers n. Thus, if G has no elements of infinite height, then $I(G) = I(B)$. Moreover, since B is a direct sum of cyclic p-groups, I(B) consists of those k for which one of the summands of B has order p^{k+1} (see [3, p. 109]). From this observation, it follows that I(G) consists of all k such that G has a cyclic direct summand of order p^{k+1}.

2.6 *Definition.* A class $\mathscr{R}$ of p-groups will be called *residual* if

(1) $\mathscr{R}$ is closed under isomorphism,

(2) $\mathscr{R}$ is closed under finite direct sums, and

(3) if $G \in \mathscr{R}$ and $X \subset I(G)$, then there exists a pure subgroup H of G such that $H \in \mathscr{R}$ and $I(H) = X$.

Examples of residual classes of p-groups are (a) the class of all direct sums of cyclic p-groups, (b) the class of all p-groups without elements of infinite height, (c) the class of all bounded p-groups, (d) the class of all closed (torsion-complete) p-groups.

2.7 *Definition.* Let $\mathscr{R}$ be a residual class of p-groups. For any p-group G, define

$$\mathfrak{I}(G; \mathscr{R}) = \{I(A) \mid A \text{ is a pure subgroup of } G \text{ and } A \in \mathscr{R}\}\,.$$

From Definitions 2.6 and 2.7 and Lemmas 2.3 and 2.5, we obtain the main result of this section.

2.8 THEOREM. $\mathfrak{I}(G; \mathscr{R})$ *is an ideal in the Boolean algebra of all subsets of* I(G).

It is obvious that $\mathfrak{I}(G; \mathscr{R})$ is invariant under isomorphism of p-groups. That is, if $G \cong H$, then $\mathfrak{I}(G; \mathscr{R}) = \mathfrak{I}(H; \mathscr{R})$.

It is useful to note a trivial consequence of Definition 2.7.

2.9 LEMMA. *If* H *is a pure subgroup of* G, *then* $\mathfrak{I}(H; \mathscr{R}) \subset \mathfrak{I}(G; \mathscr{R})$.

The particular residual class in which we are interested is the class $\mathscr{R}$ of all closed p-groups. With this understood, it is possible to write $\mathfrak{I}(G)$ instead of $\mathfrak{I}(G; \mathscr{R})$ without danger of confusion. Moreover, the only groups henceforth considered are groups without elements of infinite height.

3. FIRST EXISTENCE THEOREM

In this section, we give a method of constructing for any basic group B and any ideal $\mathscr{I}$ in P(I(B)) a group G without elements of infinite height, such that $\mathfrak{I}(G)$ is the given ideal $\mathscr{I}$. Moreover, if $\mathscr{I}$ contains all finite subsets of I(B), then B is a basic subgroup of G.

Let $B = \Sigma_{i \in \omega} \oplus B_i$, where B_i is a direct sum of cyclic groups of order p^{i+1} (or $B_i = 0$). Note that

$$I(B) = \{i \in \omega \mid B_i \neq 0\}.$$

Let $\overline{B}$ denote the closure (or torsion completion) of B, that is, the torsion subgroup of the complete direct sum $\Sigma^*_{i \in \omega} \oplus B_i$. The elements of $\overline{B}$ can be regarded as infinite sums $\Sigma_{i \in \omega} b_i$, where $b_i \in B_i$, and $O(b_i)$ (the order of b_i) has a bound independent of i. This representation of the elements of $\overline{B}$ is unique, once the decomposition $B = \Sigma \oplus B_i$ is specified. Define a mapping

$$\delta\colon \overline{B} \to P(I(B))$$

by

(1) $$\delta\left(\sum b_i\right) = \{i \in I(B) \mid b_i \neq 0\}.$$

The definition of δ naturally depends on the decomposition $B = \Sigma \oplus B_i$. However, we shall assume that a decomposition is given and fixed once and for all.

For an ideal $\mathscr{I}$ of P(I(B)), define

(2) $$\mathfrak{G}(\mathscr{I}) = \{x \in \overline{B} \mid \delta(x) \in \mathscr{I}\}.$$

3.1 THEOREM. $\mathfrak{G}(\mathscr{I})$ *is a pure subgroup of* $\overline{B}$, *and* $\mathfrak{I}(\mathfrak{G}(\mathscr{I})) = \mathscr{I}$.

Proof. If $x, y \in \mathfrak{G}(\mathscr{I})$, then

$$\delta(x - y) \subset \delta(x) \cup \delta(y) \in \mathscr{I}.$$

Hence, $x - y \in \mathfrak{G}(\mathscr{I})$. Therefore, $\mathfrak{G}(\mathscr{I})$ is a subgroup of $\overline{B}$. Suppose that $x \in \overline{B}$ and $p^k x \in \mathfrak{G}(\mathscr{I})$, that is, $\delta(p^k x) \in \mathscr{I}$. Let $x = \Sigma b_i$, where $b_i \in B_i$. Define $c_i = b_i$ if $p^k b_i \neq 0$, and $c_i = 0$ otherwise. Let $y = \Sigma c_i$. Clearly, $y \in \overline{B}$ and

$$\delta(y) = \delta(p^k x) \in \mathscr{I},$$

so that $y \in \mathfrak{G}(\mathscr{I})$. Moreover, $p^k y = p^k x$. This shows that $\mathfrak{G}(\mathscr{I})$ is pure in $\overline{B}$. For $I \in \mathscr{I}$, define

$$\overline{B}_I = \{x \in \overline{B} \mid \delta(x) \subset I\}.$$

Clearly, $\overline{B}_I \subset \mathfrak{G}(\mathscr{I})$. Moreover, $\overline{B}_I = \mathfrak{G}((I))$, where (I) denotes the principal ideal in $P(I(B))$ generated by I. Thus, $\overline{B}_I$ is a pure subgroup of $\overline{B}$, and hence it is a pure subgroup of $\mathfrak{G}(\mathscr{I})$. Moreover,

$$\overline{B}_I = \overline{B} \cap \sum_{i \in I}^* \oplus B_i$$

is the torsion subgroup of $\Sigma^*_{i\in I} \oplus B_i$. That is, $\overline{B}_I$ is the closure of $B_I = \Sigma_{i\in I} \oplus B_i$. Therefore, $I = I(B_I) = I(\overline{B}_I) \in \mathfrak{J}(\mathfrak{G}(\mathscr{I}))$. This proves that $\mathscr{I} \subset \mathfrak{J}(\mathfrak{G}(\mathscr{I}))$. To reverse the inclusion, it suffices to prove the following fact: If $I \in \mathfrak{J}(\mathfrak{G}(\mathscr{I}))$, then there exists $x \in \mathfrak{G}(\mathscr{I})$ such that $\delta(x) \supset I$. Suppose that $I = \{i_1, i_2, \cdots\}$, where $i_1 < i_2 < \cdots$. Since $I \in \mathfrak{J}(\mathfrak{G}(\mathscr{I}))$, there exists a closed pure subgroup H of $\mathfrak{G}(\mathscr{I})$ such that $I(H) = I$. Hence, for each $k \geq 1$, there exists a $y_k \in H$ such that $py_k = 0$ and $h_H(y_k) = h_{\overline{B}}(y_k) = i_k$. Consequently,

$$y_k = \sum_{j \geq i_k} b_{j,k},$$

where $b_{j,k} \in B_j$, $pb_{j,k} = 0$ for all j, and $b_{i_k,k} \neq 0$. Define the sequence $x_1, x_2, \cdots$ inductively by the rule that $x_1 = y_1$, and for $k > 1$,

$$x_k = x_{k-1} \quad \text{if } i_k \in \delta(x_{k-1}) \qquad \text{and} \qquad x_k = x_{k-1} + y_k \quad \text{if } i_k \notin \delta(x_{k-1}).$$

From this definition it is clear that $i_1, i_2, \cdots, i_k$ are in $\delta(x_k)$, and $x_1, x_2, \cdots$ converges to an element $x \in \overline{B}$. Since H is closed, $x \in H \subset \mathfrak{G}(\mathscr{I})$. Moreover, $I \subset \delta(x)$. This completes the proof of the theorem.

3.2 COROLLARY. *Let* $J = \bigcup_{I \in \mathscr{I}} I$. *Then* $\Sigma_{j\in J} \oplus B_j$ *is a basic subgroup of* $\mathfrak{G}(\mathscr{I})$.

Proof. If $x \in \Sigma_{j\in J} \oplus B_j$, then $\delta(x)$ is a finite subset of J. Hence,

$$\delta(x) \subset \bigcup_{k=1}^{n} I_k \qquad (I_k \in \mathscr{I}).$$

Therefore $\delta(x) \in \mathscr{I}$, and $x \in \mathfrak{G}(\mathscr{I})$. Thus, $\Sigma_{j\in J} \oplus B_j$ is a subgroup of $\mathfrak{G}(\mathscr{I})$. It suffices to prove that $\mathfrak{G}(\mathscr{I}) / \Sigma_{j\in J} \oplus B_j$ is divisible. Since $\overline{B}/B$ is divisible, and since if $x \in \mathfrak{G}(\mathscr{I})$ the representation $x = \Sigma_{i\in\delta(x)} b_i$ $(\delta(x) \in \mathscr{I})$ is unique, this result follows by a standard argument.

The following fact is an immediate consequence of Corollary 3.2.

3.3 COROLLARY. *If* $\mathscr{I}$ *contains all finite subsets of* I(B), *then* B *is a basic subgroup of* $\mathfrak{G}(\mathscr{I})$.

As an application of Theorem 3.1, we can give an example of two primary groups that have the same basic subgroup, are quasi-isomorphic, but are not isomorphic. (The question whether such groups exist was raised by Professor L. Fuchs at the New Mexico State University Conference on Abelian Groups in June, 1962.) For the purpose of this construction, let B be a standard basic group, that is, let $f_B(k) = 1$ for all $k \in \omega$. Let E denote the set of all finite even ordinals, and let O be the set of all finite odd ordinals. Define

$$\mathscr{I}_e = \{S \subset \omega \mid S \cap O \text{ is finite}\},$$

$$\mathscr{I}_0 = \{S \subset \omega \mid S \cap E \text{ is finite}\}.$$

Then $\mathfrak{G}(\mathscr{I}_e)$ consists of all $\Sigma_{i<\omega} b_i$ in $\overline{B}$ such that $b_i = 0$ for almost all odd i, and $\mathfrak{G}(\mathscr{I}_0)$ consists of all $\Sigma_{i<\omega} b_i$ in $\overline{B}$ such that $b_i = 0$ for almost all even i. It is obvious that $p\mathfrak{G}(\mathscr{I}_0)$ is isomorphic to $\mathfrak{G}(\mathscr{I}_e)$. Thus, $\mathfrak{G}(\mathscr{I}_0)$ and $\mathfrak{G}(\mathscr{I}_e)$ are quasi-isomorphic. However, these groups are not isomorphic, since $\mathfrak{J}(\mathfrak{G}(\mathscr{I}_0)) = \mathscr{I}_0$ and $\mathfrak{J}(\mathfrak{G}(\mathscr{I}_e)) = \mathscr{I}_e$.

4. SECOND EXISTENCE THEOREM

In the remainder of the paper, we consider only groups that have a standard basic subgroup, that is, groups G such that $f_G(k) = 1$ for all $k \in \omega$. All of the results that we obtain are valid more generally for unbounded groups G such that $f_G(k) \leq 1$ for almost all k. This generalization is trivial and only requires slightly more elaborate notation.

The following notation will henceforth be standard:

$$B = \sum_{i<\omega} \oplus \{b_i\}, \qquad O(b_i) = p^{i+1},$$

denotes a standard basic group, and $\overline{B}$ is the closure of B. The notation $\delta(x)$ and $\mathfrak{G}(\mathscr{I})$ introduced in Section 3 will refer to this decomposition of B.

The ideals in which we shall be most interested are those containing all finite subsets of ω. Such ideals will be called *free*. It follows from Corollary 3.3 that if $\mathscr{I}$ is a free ideal, then B is a basic subgroup of $\mathfrak{G}(\mathscr{I})$. In this case $\overline{B}/\mathfrak{G}(\mathscr{I})$ is divisible.

The problem to be considered is the following: For a given ideal $\mathscr{I}$, is there a pure subgroup G of $\overline{B}$ such that $\mathfrak{J}(G) = \mathscr{I}$, and $\overline{B}/G$ has a rank one? The following lemma, useful in this investigation, is well known (see [3, p. 94]).

4.1 LEMMA. *If* K *is a pure subgroup of* G, *then the natural mapping of* G[p] *into* (G/K)[p] *induces an isomorphism*

$$G[p]/K[p] \cong (G/K)[p].$$

4.2 THEOREM. *Let* $\mathscr{M}$ *be a maximal (proper) ideal of* $P(\omega)$. *Then* $\overline{B}/\mathfrak{G}(\mathscr{M})$ *has rank one.*

Proof. By Lemma 4.1, it is sufficient to prove that $\overline{B}[p]/\mathfrak{G}(\mathscr{M})[p]$ is one-dimensional. For a subset K of ω, define

$$x_K = \sum_{i \in K} p^i b_i.$$

Then every element $x \in \overline{B}[p]$ has a unique representation

$$x = x_{K_1} + 2x_{K_2} + \cdots + (p-1)x_{K_{p-1}},$$

where $K_1, K_2, \cdots, K_{p-1}$ are disjoint subsets of ω. Moreover, if $x \notin \mathfrak{G}(\mathcal{M})$, then there exists an r $(1 \leq r \leq p - 1)$ such that $K_r \notin \mathcal{M}$. Suppose that x and y are in $\overline{B}[p]$ and not in $\mathfrak{G}(\mathcal{M})$. Let

$$x = x_{K_1} + 2x_{K_2} + \cdots + (p - 1)x_{K_{p-1}},$$

$$y = x_{L_1} + 2x_{L_2} + \cdots + (p - 1)x_{L_{p-1}}.$$

Suppose that $K_r \notin \mathcal{M}$ and $L_s \notin \mathcal{M}$. Choose an integer t such that $rt \equiv s \pmod p$. Then

$$K_r \cap L_s \cap \delta(tx - y) = \emptyset.$$

Since $\mathcal{M}$ is maximal, it follows that $\delta(tx - y) \in \mathcal{M}$ and $tx - y \in \mathfrak{G}(\mathcal{M})[p]$. This shows that $\overline{B}[p]/\mathfrak{G}(\mathcal{M})[p]$ is one-dimensional and proves the theorem. (The group $G(\mathcal{M})$ can be considered as the torsion subgroup of an ultraproduct of the cyclic factors of B (see [2]). It is easy to give a proof of Theorem 4.2 based on the fundamental theorem of the first-order language of ultraproducts (Theorem 2.2 of [2]).)

It follows in particular from Theorem 4.2 that there are 2^c nonisomorphic pure subgroups G of $\overline{B}$ such that $B \subset G$ and $\overline{B}/G \cong Z(p^\infty)$, since there are 2^c free, maximal ideals in $P(\omega)$ (see [4]), and if $\mathcal{M}$ and $\mathcal{N}$ are distinct, free, maximal ideals, then $\mathfrak{G}(\mathcal{M})$ is not isomorphic to $\mathfrak{G}(\mathcal{N})$ (because $\mathfrak{J}(\mathfrak{G}(\mathcal{M})) = \mathcal{M} \neq \mathcal{N} = \mathfrak{J}(\mathfrak{G}(\mathcal{N}))$).

We shall prove shortly that for a group of the form $\mathfrak{G}(\mathcal{I})$, $\overline{B}/\mathfrak{G}(\mathcal{I})$ has rank one only if $\mathcal{I}$ is a maximal ideal. To obtain this result we use a special case of a lemma to be used in the proof of the main result of the next section.

4.3 LEMMA. *Let* A *be a pure, closed subgroup of* $\overline{B}$. *Let* $z \in \overline{B}[p]$. *Then there exists an* $x \in A[p]$ *such that* $\delta(z - x) \cap I(A) = \emptyset$.

Proof. Let $I(A) = \{i_1, i_2, \cdots\}$, where $i_1 < i_2 < \cdots$. By Lemma 2.2, there exist $y_1, y_2, \cdots$ in $A[p]$ such that $h_{\overline{B}}(y_k) = h_A(y_k) = i_k$. Consequently,

$$y_k = \sum_{j \geq i_k} u_{j,k} p^j b_j,$$

where $u_{j,k}$ is an integer and p does not divide $u_{i_k,k}$. Without loss of generality, it can be assumed that $u_{i_k,k} = 1$. Suppose that

$$z = \sum_{j \in \omega} v_j p^j b_j,$$

where v_j is an integer. A sequence $x_1, x_2, \cdots$ of elements of $A[p]$ can be defined by induction such that

$$x_k = \sum_{j \in \omega} w_{j,k} p^j b_j,$$

where the $w_{j,k}$ are integers such that $w_{j,\ell} = w_{j,k}$ for $k \leq \ell$ and $j \leq i_k$, and

$$w_{i_j,k} = v_{i_j} \quad (j \leq k).$$

It is clear that the sequence $x_1, x_2, \cdots$ converges in the p-adic topology of $\overline{B}$ to an element x of order p. Since A is closed, $x \in A$. If

$$x = \sum_{j\in\omega} w_j p^j b_j ,$$

then $w_{i_k} = v_{i_k}$ for all k. Hence, $\delta(z - x) \cap I(A) = \emptyset$.

4.4 LEMMA. *Let* $\mathscr{I}$ *be an ideal in* $P(\omega)$. *Let* A *be a pure, closed subgroup of* $\overline{B}$. *Define* $\mathscr{J} = \{R \cap I(A) \mid R \in \mathscr{I}\}$, *so that* $\mathscr{J}$ *is an ideal of* $P(I(A))$. *Then*

$$|(A[p] + \mathfrak{S}(\mathscr{I})[p]) / \mathfrak{S}(\mathscr{I})[p]| \geq |P(I(A)) / \mathscr{J}|.$$

Proof. Let $\{R_t \mid t \in T\}$ be a set of representatives of the classes of $P(I(A))/\mathscr{J}$. Thus, $R_t \subset I(A)$ for all t, and if $s \neq t$, then

$$[R_t \cap (I(A) - R_s)] \cup [R_s \cap (I(A) - R_t)] \notin \mathscr{J} .$$

Moreover, $|T| = |P(I(A))/\mathscr{J}|$. By Lemma 4.3 there exists, for each $t \in T$, an element $x_t \in A[p]$ such that $\delta(x_t) \cap I(A) = R_t$. If $t \neq s$, then

$$\delta(x_s - x_t) \supset [R_t \cap (I(A) - R_s)] \cup [R_s \cap (I(A) - R_t)].$$

Therefore, $\delta(x_s - x_t) \notin \mathscr{I}$, and $x_s - x_t \notin \mathfrak{S}(\mathscr{I})$. This shows that

$$|(A[p] + \mathfrak{S}(\mathscr{I})[p]) / \mathfrak{S}(\mathscr{I})[p]| \geq |T| ,$$

which proves the lemma.

In the proof of Lemma 4.4, if $s \neq t$ and R_s and R_t do not belong to $\mathscr{J}$, then $\delta(ux_s - vx_t) \notin \mathscr{I}$ unless $ux_s = vx_t = 0$. From this remark, it follows that if $P(I(A))/\mathscr{J}$ contains more than two elements, then $(A[p] + \mathfrak{S}(\mathscr{I})[p]) / \mathfrak{S}(\mathscr{I})[p]$ is not one-dimensional.

4.5 THEOREM. *Let* $\mathscr{I}$ *be an ideal of* $P(\omega)$ *such that the rank of* $\overline{B}/\mathfrak{S}(\mathscr{I})$ *is one. Then* $\mathscr{I}$ *is a maximal proper ideal of* $P(\omega)$.

Proof. If $\mathscr{I}$ is not maximal, then $P(\omega)/\mathscr{I}$ contains at least two nonzero elements. In this case, $\overline{B}[p]/\mathfrak{S}(\mathscr{I})[p]$ has dimension greater than one. Thus, by Lemma 4.1, the rank of $\overline{B}/\mathfrak{S}(\mathscr{I})$ (which equals the dimension of $(\overline{B}/\mathfrak{S}(\mathscr{I}))[p]$) is greater than one.

5. THIRD EXISTENCE THEOREM

We return to the problem of constructing pure subgroups G of $\overline{B}$ for which $\overline{B}/G$ has rank one and $\mathfrak{I}(G)$ is a prescribed ideal. The notation of the last section is continued. In particular, B denotes a standard basic group.

5.1 LEMMA. *Let* $B \subset H \subseteq \overline{B}$. *Let* $P \subset \overline{B}[p]$ *be such that* $H[p] \subset P$. *Then there exists a pure subgroup* G *of* $\overline{B}$ *such that* $H \subset G$ *and* $G[p] = P$.

Proof. Let π be the natural homomorphism of $\overline{B}$ onto $\overline{B}/B$. Let $K = \pi(P + H)$. There exists a subgroup L of $\overline{B}/B$ such that L is divisible, $L \supset K$, and $L[p] = K[p]$ (see [3, p. 66]). Let $G = \pi^{-1}(L)$. Clearly, G is pure in $\overline{B}$, $G \supset H$, and $G[p] \supset P$. If $x \in G[p]$, then $\pi(x) \in L[p] = K[p]$. Hence, $\pi(x) = \pi(y) + \pi(z)$, where $y \in P$, $z \in H$.

Consequently, $x - (y + z) = w \in B \subset H$, so that $x - y = z + w \in H$. Moreover, $p(z + w) = p(x - y) = 0$. Thus, $z + w \in H[p] \subset P$. Therefore, $x \in P$.

The following lemma is of some interest in its own right. The proof uses a technique introduced to abelian group theory by Crawley in [1].

5.2 LEMMA. *Let* V *be a vector space over a field* F. *Suppose that* α *is an infinite cardinal number. Let* $\{W_\xi \mid \xi < \beta\}$ *be a family of subspaces of* V *indexed by the cardinal number* $\beta \leq \alpha$, *such that* $\dim W_\xi \geq \alpha$ *for all* $\xi < \beta$. *Then there exists a subspace* U *of* V *such that* $U \not\supset W_\xi$ *for all* $\xi < \beta$, *and* V/U *is one-dimensional.*

Proof. Let $e \neq 0$ be in V. We prove by induction the existence of a sequence $\{x_\xi \mid \xi < \beta\}$ of elements of V with $x_\xi \in W_\xi$, such that the space U_β spanned by $\{e - x_\xi \mid \xi < \beta\}$ does not contain e. Choose $x_0 \in W_0$ so that $\{x_0, e\}$ is linearly independent. Then the space U_1 spanned by $e - x_0$ does not contain e. Assume that the elements x_η have been specified for $\eta < \xi$ (where ξ is an ordinal less than β) in such a way that the spaces spanned by $\{e - x_\eta \mid \eta \leq \zeta\}$ $(\zeta < \xi)$ do not contain e. Then the space U_ξ spanned by $\{e - x_\eta \mid \eta < \xi\}$ does not contain e. Moreover, the space $U_\xi + Fe$ has dimension at most $|\xi| + 1$. Since β is a cardinal number and $\xi < \beta$, it follows that

$$|\xi| + 1 < \beta + 1 \leq \alpha + 1 = \alpha \leq \dim W_\xi .$$

Hence $U_\xi + Fe \not\supset W_\xi$. Choose x_ξ in W_ξ so that $x_\xi \notin U_\xi + Fe$. Then

$$e \notin U_\xi + F(e - x_\xi) .$$

Consequently x_ξ satisfies the conditions of the inductive definition. Let U_β be the subspace of V spanned by $\{e - x_\xi \mid \xi < \beta\}$. Then $e \notin U_\beta$, while $e - x_\xi \in U_\beta$ for all $\xi < \beta$. Let U be a subspace of V that is maximal with respect to the properties $U \supset U_\beta$ and $e \notin U$. Then V/U is one-dimensional and $x_\xi \notin U$ for all ξ (since otherwise $e = x_\xi + (e - x_\xi) \in U$). Thus, $U \not\supset W_\xi$ for all $\xi < \beta$.

5.3 COROLLARY. *Let* V *be a vector space over the field* F. *Suppose that* α *is an infinite cardinal number. Let* U_0 *be a subspace of* V, *and let* $\{W_\xi \mid \xi < \beta\}$ *be a family of subspaces of* V *indexed by the cardinal number* $\beta \leq \alpha$, *such that* $\dim [(W_\xi + U_0)/U_0] \geq \alpha$ *for all* $\xi < \beta$. *Then there exists a subspace* U *of* V *such that* $U \supset U_0$, $U \not\supset W_\xi$ *for all* $\xi < \beta$, *and* V/U *is one-dimensional.*

5.4 THEOREM. *Let* $\mathscr{I}$ *be a free ideal in* $P(\omega)$ *such that every nonzero principal ideal of* $P(\omega)/\mathscr{I}$ *has the cardinal number* $\mathfrak{c}$. *Then there exists a pure subgroup* G *of* $\overline{B}$ *such that* $B \subset G$, $\overline{B}/G \cong Z(p^\infty)$, *and* $\mathfrak{I}(G) = \mathscr{I}$.

Proof. Apply Corollary 5.3, where $V = \overline{B}[p]$, F is the prime field of characteristic p, $\alpha = \mathfrak{c}$, $U_0 = \mathfrak{G}(\mathscr{I})[p]$, and the family $\{W_\xi \mid \xi < \beta\}$ consists of all subspaces of $\overline{B}[p]$ of the form $A[p]$ (A a pure, closed subgroup of $\overline{B}$ such that $I(A) \notin \mathscr{I}$). To show that the hypotheses of Corollary 5.3 are satisfied, observe that the cardinal number of the set of all pure, closed subgroups of $\overline{B}$ is at most $\mathfrak{c}$ (since each one is uniquely determined by a countable subgroup of $\overline{B}$ and there are no more than $\mathfrak{c}$ countable subsets of $\overline{B}$). Consequently, it is possible to index the set of all such $A[p]$ by a cardinal number $\beta \leq \mathfrak{c}$. By Lemma 4.4, if A is a pure, closed subgroup of $\overline{B}$, then

$$|(A[p] + \mathfrak{G}(\mathscr{I})[p]) / \mathfrak{G}(\mathscr{I})[p]| \geq |P(I(A))/\mathscr{I}| ,$$

where $\mathscr{J} = \{I(A) \cap R \mid R \in \mathscr{I}\}$. Since $I(A) \notin \mathscr{I}$, it follows that $P(I(A))/\mathscr{J}$ is isomorphic to a nonzero principal ideal of $P(\omega)/\mathscr{I}$, so that by assumption its cardinal number is $\mathfrak{c}$. Consequently,

$$\dim[(W_\xi + U_0)/U_0] = |(W_\xi + U_0)/U_0| = \mathfrak{c}.$$

Thus, the hypotheses of Corollary 5.3 are all satisfied, and there exists a subspace U of $\overline{B}[p]$ with $\mathfrak{G}(\mathscr{I})[p] \subset U$, $\dim[\overline{B}[p]/U] = 1$, and $U \not\supset A[p]$ if A is a pure, closed subgroup of $\overline{B}$ such that $I(A) \notin \mathscr{I}$. By Lemma 5.1, there exists a pure subgroup G of $\overline{B}$ such that $B \subset \mathfrak{G}(\mathscr{I}) \subset G$ and $G[p] = U$. By Lemma 4.1 and the fact that $\mathscr{I}$ is a free ideal, $\overline{B}/G \cong Z(p^\infty)$. By Lemma 2.9 and Theorem 3.1,

$$\mathscr{I} = \mathfrak{I}(\mathfrak{G}(\mathscr{I})) \subset \mathfrak{I}(G).$$

On the other hand, if A is a pure, closed subgroup of G, then $I(A) \in \mathscr{I}$. For if $I(A) \notin \mathscr{I}$, then $G[p] = U \not\supset A[p]$, which is a contradiction. Consequently, $\mathfrak{I}(G) = \mathscr{I}$.

It is natural to ask what ideals satisfy the condition of Theorem 5.4; that is, for what ideals $\mathscr{I}$ of $P(\omega)$ is it true that every nonzero principal ideal of $P(\omega)/\mathscr{I}$ has cardinality $\mathfrak{c}$? Using some known results on Stone-Čech compactification, we can show that this condition is equivalent to the fact that $P(\omega)/\mathscr{I}$ has no atoms.

It is known (see [4, p. 133]) that if N is a countably discrete space, and A is an infinite closed subset of $\beta N - N$ (where βN is the Stone-Čech compactification of N), then there exists a closed subspace B of A such that B is homeomorphic to βN. If this fact is translated into algebraic terms, it is equivalent to the statement that if $\mathscr{J}$ is any free ideal in $P(\omega)$ such that $P(\omega)/\mathscr{J}$ is infinite, then there exists an ideal $\mathscr{K} \supset \mathscr{J}$ such that $P(\omega)/\mathscr{K}$ is isomorphic to $P(\omega)$. In particular, it follows that $|P(\omega)/\mathscr{J}| = \mathfrak{c}$. If $P(\omega)/\mathscr{I}$ has no atoms, then every nonzero principal ideal of $P(\omega)/\mathscr{I}$ is infinite. Since every such principal ideal is isomorphic as a Boolean algebra to $P(\omega)/\mathscr{J}$ for a suitable ideal $\mathscr{J} \supset \mathscr{I}$, it follows that the nonzero principal ideals of $P(\omega)/\mathscr{I}$ have cardinality $\mathfrak{c}$. Conversely, it is obvious that if every nonzero principal ideal of $P(\omega)/\mathscr{I}$ has cardinality $\mathfrak{c}$, then $P(\omega)/\mathscr{I}$ has no atoms.

5.5 COROLLARY. *Let* $\mathscr{I}$ *be a free ideal in* $P(\omega)$ *such that* $P(\omega)/\mathscr{I}$ *has no atoms. Then there exists a pure subgroup* G *of* $\overline{B}$ *with* $B \subset G$ *such that* $\overline{B}/G \cong Z(p^\infty)$ *and* $\mathfrak{I}(G) = \mathscr{I}$.

A particularly interesting case in which $P(\omega)/\mathscr{I}$ has no atoms is that where $\mathscr{I}$ is the ideal of all finite subsets of ω. In this case the group G satisfying the conditions $B \subset G \subset \overline{B}$, G pure in $\overline{B}$, $\overline{B}/G \cong Z(p^\infty)$, $\mathfrak{I}(G) = \mathscr{I}$ has the property that it is almost indecomposable. Specifically, if $G = H \oplus K$, then either H or K is finite. This follows from an observation of Crawley that if G is a pure subgroup of $\overline{B}$ such that $\overline{B}/G$ has rank one, then for any direct decomposition $G = H \oplus K$, either H or K is closed. Indeed,

$$(\overline{H} \oplus \overline{K})/G \cong \overline{H}/H \oplus \overline{K}/K,$$

so that either $H = \overline{H}$ or $K = \overline{K}$. If also $\mathfrak{I}(G)$ consists of only finite sets, then every pure, closed subgroup of G is finite. Groups with this "indecomposability" property have been constructed previously by Crawley and by Pierce (see [7, p. 308]). (The construction given by Crawley in [1] to obtain a pure subgroup G of B having no isomorphic proper subgroups can easily be modified so that the group G also satisfies the condition $\overline{B}/G \cong Z(p^\infty)$. Such a group G cannot have an infinite closed direct summand, since every infinite closed group contains a subgroup isomorphic to $\overline{B}$. This observation is due to John Irwin.)

6. AN EXAMPLE

Many ideals $\mathscr{I}$ in $P(\omega)$ fail to satisfy the condition that $P(\omega)/\mathscr{I}$ has no atoms. For such ideals, we do not know whether there exist pure subgroups G of $\overline{B}$ (where B is standard) such that $\overline{B}/G \cong Z(p^\infty)$ and $\mathfrak{J}(G) = \mathscr{I}$. Particularly interesting is the case in which $\mathscr{I}$ is $\mathscr{M} \cap \mathscr{N}$, the intersection of two maximal ideals. For such ideals, the method used to prove Theorem 5.4 may break down in an essential way. The most important feature of the proof of Theorem 5.4 is the construction of a vector space U between $\mathfrak{S}(\mathscr{I})[p]$ and $\overline{B}[p]$ such that $A[p] \not\subset U$ for every pure, closed subgroup A of $\overline{B}$ with $I(A) \notin \mathscr{I}$. It is easy to see that if $\mathscr{I}$ is the intersection of two distinct maximal ideals, then $\overline{B}[p]/\mathfrak{S}(\mathscr{I})[p]$ is two-dimensional. Thus it is not surprising that an extension U of $\mathfrak{S}(\mathscr{I})[p]$, having the property $A[p] \not\subset U$ for all pure, closed $A \subset \overline{B}$ with $I(A) \notin \mathscr{I}$, may not exist in this case. In this section we give an example of such an ideal $\mathscr{I}$ and show that, nevertheless, for this particular ideal there exists an extension G of $\mathfrak{S}(\mathscr{I})$ that is pure in $\overline{B}$ and satisfies $\overline{B}/G \cong Z(p^\infty)$ and $\mathfrak{J}(G) = \mathscr{I}$. Thus, although it is possible that for any free ideal $\mathscr{I}$ there exists a pure subgroup G of $\overline{B}$ such that $\overline{B}/G \cong Z(p^\infty)$ and $\mathfrak{J}(G) = \mathscr{I}$, such a group cannot always be constructed by suitably extending the socle of $\mathfrak{S}(\mathscr{I})$. The following example is a modification of a group constructed by Hill (see [5, p. 311]).

Example. Let $E = \{2n \mid n \in \omega\}$, $O = \{2n + 1 \mid n \in \omega\}$. Let $\mathscr{N}_1$ be a free maximal ideal in P(E) such that

$$\{n \mid n \equiv 2 \pmod 4\} \in \mathscr{N}_1.$$

Define $\mathscr{N}_2 = \{A' \mid A \in \mathscr{N}_1\}$, where $A' = \{n - 3 \mid n \in A\} \cap \omega$. Then $\mathscr{N}_2$ is a free maximal ideal in P(O) such that

$$\{n \mid n \equiv 3 \pmod 4\} \in \mathscr{N}_2.$$

Let

$$\mathscr{I} = \{A \cup B \mid A \in \mathscr{N}_1,\ B \in \mathscr{N}_2\}.$$

Then $\mathscr{I}$ is a free ideal in $P(\omega)$, and $\mathscr{I}$ is the intersection of the two maximal ideals $(\mathscr{N}_1, O)$ and $(\mathscr{N}_2, E)$. Let

$$B = \sum_{i<\omega} \oplus \{b_i\}, \quad \text{where } O(b_i) = p^{i+1}.$$

As in Section 4, if $S \subset \omega$, we denote by x_S the element $\Sigma_{i \in S}\, p^i b_i$ of $\overline{B}[p]$.

Suppose that $\mathfrak{S}(\mathscr{I})[p]$ is properly contained in $U \subset \overline{B}[p]$. Then there exist disjoint sets $S_1, S_2, \cdots, S_{p-1} \subset E$, and $T_1, T_2, \cdots, T_{p-1} \subset O$ such that

$$(x_{S_1} + x_{T_1}) + 2(x_{S_2} + x_{T_2}) + \cdots + (p-1)(x_{S_{p-1}} + x_{T_{p-1}}) \in U,$$

and

$$S_1 \cup S_2 \cup \cdots \cup S_{p-1} \cup T_1 \cup T_2 \cup \cdots \cup T_{p-1} \notin \mathscr{I}.$$

For convenience, let

$$S_0 = E - (S_1 \cup S_2 \cup \cdots \cup S_{p-1}) \quad \text{and} \quad T_0 = O - (T_1 \cup T_2 \cup \cdots \cup T_{p-1}).$$

Then there exists exactly one j $(0 \leq j \leq p - 1)$ and exactly one k $(0 \leq k \leq p - 1)$ such that $S_j \notin \mathscr{I}$ and $T_k \notin \mathscr{I}$. Moreover, j and k are not both zero. It follows that $jx_E + kx_O \in U$, where j and k are not both zero. If $k = 0$, then $j \neq 0$ and it is easy to see that U contains x_S for all $S \subset E$. Consequently, $U \supset \overline{B}_e[p]$, where $\overline{B}_e$ is the pure, closed subgroup of $\overline{B}$ that is the closure of the subgroup generated by $\{b_{2n} \mid n < \omega\}$. Evidently, $I(\overline{B}_e) = E \notin \mathscr{I}$. Thus, we can suppose that $k \neq 0$. Let A be the closure in $\overline{B}$ of the subgroup K of B generated by $\{jp^3 b_{2n} + kb_{2n-3} \mid n \geq 2\}$. Then A is pure in $\overline{B}$, since K is pure in B. Moreover, K is a basic subgroup of A, and since $0 < k < p$, it follows that $I(A) = O \notin \mathscr{I}$. Notice that every element of $A[p]$ is of the form

$$(jx_{S_1} + kx_{S_1'}) + 2(jx_{S_2} + kx_{S_2'}) + \cdots + (p - 1)(jx_{S_{p-1}} + kx_{S_{p-1}'}),$$

where $S_1, S_2, \cdots, S_{p-1}$ are disjoint subsets of E and $S_i' = \{n - 3 \mid n \in S_i\}$. If $S_i \in \mathscr{N}_1$, then $S_i' \in \mathscr{N}_2$, so that

$$jx_{S_i} + kx_{S_i'} \in \mathfrak{G}(\mathscr{I})[p] \subset U.$$

If $S_i \notin \mathscr{N}_1$, then $E - S_i \in \mathscr{N}_1$, and $O - S_i' = (E - S_i)' \in \mathscr{N}_2$. Therefore,

$$jx_{S_i} + kx_{S_i'} = (jx_E + kx_O) - (jx_{E-S_i} + kx_{O-S_i'}) \in U + \mathfrak{G}(\mathscr{I})[p] = U.$$

Consequently, $A[p] \subset U$. To summarize, we have shown that for any group U with $\mathfrak{G}(\mathscr{I})[p]$ properly contained in $U \subset \overline{B}[p]$, there exists a pure, closed subgroup A of $\overline{B}$ with $I(A) \notin \mathscr{I}$, such that $A[p] \subset U$. The rest of the example will be devoted to constructing a pure subgroup G of $\overline{B}$ with $\mathfrak{G}(\mathscr{I}) \subset G$, satisfying $\overline{B}/G \cong Z(p^\infty)$ and $\mathfrak{I}(G) = \mathscr{I}$.

For $n \geq 1$, define

$$u_n = \sum_{k \geq n} p^{4(k-n)} (b_{4k-3} + pb_{4k}).$$

Note that $O(u_n) = p^{4n}$, and

$$u_n - p^4 u_{n+1} = b_{4n-3} + pb_{4n} \in B \subset \mathfrak{G}(\mathscr{I}). \tag{1}$$

Moreover, since $\delta(p^3 u_1) = \{4k \mid k \geq 1\}$, it follows that $p^3 u_1 \notin \mathfrak{G}(\mathscr{I})$. Define G to be the subgroup of $\overline{B}$ generated by $\{\mathfrak{G}(\mathscr{I}), u_1, u_2, \cdots\}$. It follows from (1) that $G/\mathfrak{G}(\mathscr{I}) \cong Z(p^\infty)$. Hence, G is pure in $\overline{B}$. Moreover,

$$\begin{aligned}\overline{B}/\mathfrak{G}(\mathscr{I}) &\cong \overline{B}_E/\overline{B}_E \cap \mathfrak{G}(\mathscr{I}) \oplus \overline{B}_O/\overline{B}_O \cap \mathfrak{G}(\mathscr{I}) \\ &= \overline{B}_E/\mathfrak{G}_E(\mathscr{N}_1) \oplus \overline{B}_O/\mathfrak{G}_O(\mathscr{N}_2) \cong Z(p^\infty) \oplus Z(p^\infty)\end{aligned}$$

by Lemma 4.3. ($\overline{B}_E$ is the closure of $\Sigma_{i \in E} \oplus \{b_i\}$, $\mathfrak{G}_E(\mathscr{N}_1)$ is the pure subgroup of $\overline{B}_E$ constructed from the ideal $\mathscr{N}_1$ by a definition analagous to (2) of Section 3, and $\overline{B}_O$ and $\mathfrak{G}_O(\mathscr{N}_2)$ are similarly defined.) It follows that $\overline{B}/G \cong Z(p^\infty)$. Since $\mathfrak{G}(\mathscr{I})$ is a pure subgroup of G, we have $\mathscr{I} = \mathfrak{I}(\mathfrak{G}(\mathscr{I})) \subset \mathfrak{I}(G)$. To show that $\mathfrak{I}(G) = \mathscr{I}$, it is sufficient to prove that $O \notin \mathfrak{I}(G)$ and $E \notin \mathfrak{I}(G)$.

Suppose that $O \in \mathfrak{I}(G)$. Then by Lemma 4.3, there exists an $x \in G[p]$ satisfying $O \subset \delta(x)$. By (1) we can write $x = z + ru_n$, where $z \in \mathfrak{G}(\mathscr{I})$, $n \geq 1$, and r is an integer not divisible by p^4. Since $px = 0$, we see that $rpu_n = -pz \in \mathfrak{S}(\mathscr{I})$. Hence by (1), $rpu_n = 0$. Therefore, $n = 1$ and p^3 divides r. Write $r = p^3 s$, where p does not divide s. Then

$$x = z + s \sum_{k \geq 1} p^{4k-1}(b_{4k-3} + pb_{4k}) = z + s \sum_{k \geq 1} p^{4k} b_{4k}.$$

Thus, $O \subset \delta(x) \subset \delta(z) \cup \{4k \mid k \geq 1\}$. This implies that $O \subset \delta(z) \in \mathscr{I}$, which is a contradiction. Consequently, $O \notin \mathfrak{I}(G)$.

Suppose that $E \in \mathfrak{I}(G)$. Let H be a pure, closed subgroup of G such that $I(H) = E$. Then there exist elements $w_n \in H$ of the form

$$w_n = \sum_{k < 4n} a_{k,n} b_k + b_{4n} + \sum_{k > 4n} a_{k,n} p^{k-4n} b_k$$

such that the closure in $\overline{B}$ of the group generated by the w_n for $n \geq 1$ is contained in H. Note that

$$p^{4n-2} w_n = a_{4n-2,n} p^{4n-2} b_{4n-2} + a_{4n-1,n} p^{4n-2} b_{4n-1} + p^{4n-2} b_{4n} + \sum_{k > 4n} a_{k,n} p^{k-2} b_k.$$

By induction it is possible to construct a sum of the elements $p^{4n-2} w_n$ that has the form

$$\sum_{n \geq 2} (c_n p^{4n-5} b_{4n-3} + e_n p^{4n-4} b_{4n-2} + f_n p^{4n-3} b_{4n-1} + d_n p^{4n-2} b_{4n}),$$

where

(2)
$$d_n = 1 \quad \text{if } c_n \equiv 0 \pmod{p^3}, \text{ and}$$
$$d_n = 0 \quad \text{otherwise.}$$

Since $\mathscr{I}$ contains the set $\{n \mid n \equiv 2 \pmod 4\} \cup \{n \mid n \equiv 3 \pmod 4\}$, it follows that there exists an element

(3)
$$x = \sum_{n \geq 2} (c_n p^{4n-5} b_{4n-3} + d_n p^{4n-2} b_{4n})$$

with the c_n and d_n related by (2), such that $x \in H + \mathfrak{S}(\mathscr{I}) \subset G$. Note that $O(x) \leq p^3$. By an argument like the one given in the previous paragraph, this means that x can be represented in the form $x = z + rpu_1$ ($z \in \mathfrak{S}(\mathscr{I})$). Thus, we have the identity

(4)
$$z + \sum_{n \geq 1} (r p^{4n-3} b_{4n-3} + s p^{4n-2} b_{4n}) = \sum_{n \geq 2} (c_n p^{4n-5} b_{4n-3} + d_n p^{4n-2} b_{4n}).$$

Let $\delta(x) = A \cup B$, where $A \in \mathscr{N}_1$, $B \in \mathscr{N}_2$. By the definition of $\mathscr{N}_2$, it follows that $A' \in \mathscr{N}_2$. Therefore, $A' \cup B \in \mathscr{N}_2$. Since $\{k \in \omega \mid k \equiv 1 \pmod 4\} \notin \mathscr{N}_2$, there exists an $m \in \omega$ such that $m \notin A' \cup B$ and $m \equiv 1 \pmod 4$. Let $n = (1/4)(m + 3)$. Then $4n \notin A$ and $4n - 3 \notin B$. Hence by (4),

(5) $$r p^{4n-3} b_{4n-3} = c_n p^{4n-5} b_{4n-3},$$

(6) $$r p^{4n-2} b_{4n} = d_n p^{4n-2} b_{4n}.$$

By (2), if p^3 divides c_n, then $d_n = 1$. Therefore, $r \equiv 1 \pmod{p^3}$, by (6). However, it then follows from (5) that p^3 cannot divide c_n. On the other hand, if p^3 does not divide c_n, then $d_n = 0$. Thus, by (6), p^3 divides r. Then by (5) $c_n p^{4n-5} b_{4n-3} = 0$, so that p^3 divides c_n. This contradiction proves that $E \notin \mathfrak{I}(G)$. Therefore, $\mathfrak{I}(G) = \mathscr{I}$.

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