CHARACTERIZATION OF n-SPHERES BY AN EXCLUDED MIDDLE MEMBRANE PRINCIPLE

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Many authors have obtained characterizations of n-spheres, especially for n = 1; Zippin in [13] gave one for n = 2 (see [11, p. 88]); Bing in [1] gave characterizations for n = 1, 2, and 3; also, in a recent paper [5], Doyle and Hocking have achieved a characterization for all n-spheres. It is our purpose to approach the problem in a somewhat different fashion. We use primarily well-known results of point-set topology; we also use results based on the set-valued set-function T, with its origins in Jones' aposyndetic [7], which has been used in studying semigroups in [3] and elsewhere.

We prove a general theorem that characterizes n-spheres for $n > 1$; because 0-spheres are not connected, we must revert to a slightly different hypothesis for the case n = 1. But both instances involve an excluded middle irreducible membrane principle that we first discovered for the case of the simple closed curve.

1. CHARACTERIZATION OF THE 1-SPHERE (SIMPLE CLOSED CURVE)

We define the set-valued set-function T, for $A \subset S$, as follows. Let Q be an open subset of S, and let W be a subcontinuum of S. Then

$$y \notin T(A) \text{ if and only if there exist } Q \text{ and } W \text{ such that } y \in Q \subset W \subset S - A.$$

For examples and fundamental properties, see [4].

Let S be an irreducible continuum [9, p. 14]; we follow Kuratowski [8] and let

$$I(b, S) = \{x\colon S \text{ is an irreducible continuum from } x \text{ to } b\}.$$

We denote the closure of A by cl(A).

The following two theorems will be assumed, under the hypothesis that X is a compact Hausdorff continuum.

(A) *For* $p \in X$, *the set* T(p) *is a continuum.* See Theorem 3 in [7] or Corollary 1.1 in [4, p. 115].

(B) *If* X *is irreducible from* a *to* b, *then* T(a) = cl(I(b, X)); *if* T(a) *has vacuous interior, then* T(a) *is a* C-*set; if* T(a) *has nonvacuous interior, then* T(a) *is an indecomposable continuum.* See Theorems 12 and 13 in [3, p. 272]. For Wallace's definition of C-set see [10, p. 639].

THEOREM 1. *Let* X *be a nondegenerate, compact, perfectly separable Hausdorff continuum. Then* X *is a* 1-*sphere if and only if for every pair* $x, y \in X$ *with*

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$x \neq y$, *there exist irreducible continua* S_1 *and* S_2 *from* x *to* y, *such that* $S_1 \neq S_2$, $X = S_1 + S_2$, *and if* S_3 *is a continuum irreducible from* x *to* y, *then either* $S_3 = S_1$ *or* $S_3 = S_2$.

Proof. The necessity follows immediately.

For the sufficiency, we first show that X contains no indecomposable continuum. Suppose N is an indecomposable subcontinuum of X. Let x and y be in the same composant C' [9, p. 57] of N. By Theorem 43 in [9, p. 15], there exists a continuum C_1 in C' irreducible from x to y. By hypothesis there exists a continuum C_2, irreducible from x to y, and such that $X = C_1 + C_2$, and so C_2 contains the nonvacuous set $X - C_1$. Since $C_2 \supset N - C_1$, it follows from the properties of an indecomposable continuum [9, Theorem 137, p. 58] that $C_2 \supset C_1$. Then $C_2 = C_1$, which contradicts the hypothesis that $C_2 \neq C_1$. Hence X does not contain an indecomposable subcontinuum.

Now let $a, b \in X$, $a \neq b$, and let S_1 and S_2 be the irreducible subcontinua from a to b in the theorem. For the space S_1, we write $T(a, S_1)$ for T(a); we wish now to prove $T(a, S_1) = a$. By (B) above and by the previous argument, $T(a, S_1)$ has vacuous interior with respect to S_1, and therefore $T(a, S_1) \neq S_1$. Now suppose $a \neq c$ and $c \in T(a, S_1)$; by (A) above, $T(a, S_1)$ is a continuum. Let N_1 be an irreducible subcontinuum in $T(a, S_1)$ from a to c; by hypothesis, there exists an irreducible subcontinuum N_2 of X such that $X = N_1 + N_2$ and $N_1 \neq N_2$. Then

$$N_2 \supset \mathrm{cl}\,(X - N_1) \supset \mathrm{cl}\,(S_1 - N_1) \supset \mathrm{cl}\,(S_1 - T(a, S_1)) = S_1 ,$$

and so $N_2 \supset N_1$. Then $N_2 = N_1$, which is a contradiction.

Hence we have shown that $T(a, S_1) = a$; similarly, $T(a, S_2) = a$ and

$$T(b, S_1) = b = T(b, S_2).$$

We use this now to show that $S_1 \cap S_2 = a + b$.

Suppose that $d \in (S_1 \cap S_2) - a - b$. By the argument above, $d \notin I(b, S_i) + I(a, S_i)$, that is, d is not an endpoint of $S_i (i = 1, 2)$. Hence some proper subcontinuum S'_i of S_i is irreducible from d to b, and therefore $a \notin S'_i$. By hypothesis, there must then exist a subcontinuum S'_3 irreducible from d to b such that $X = S'_2 + S'_3$, and so $S'_3 \supset a$; hence $S'_3 \neq S'_1$. This implies that $S'_1 = S'_2$, by our hypothesis; hence, $q \notin S'_3$ for some $q \in S'_1 \cap S'_2$. But then $S'_3 \supset a + b + d$, and there exists an S_3 in S'_3 that is a continuum irreducible from a to b. Thus $q \notin S_3$ and $q \in S_1 \cap S_2$; hence $S_1 \neq S_3$ and $S_2 \neq S_3$, which again contradicts our hypothesis. Therefore we have shown that $S_1 \cap S_2 = a + b$. Hence every pair of points separates the space, and therefore X is a 1-sphere.

We note that we need perfect separability only for showing that X is metrizable and therefore homeomorphic with the unit circle. For similar reasons we include perfect separability in the hypothesis of Theorem 2.

2. CHARACTERIZATION OF THE n-SPHERE FOR $n > 1$

Let F be a compact, perfectly separable Hausdorff space, let A(n) be a nonvoid collection of (n - 1)-spheres in F, and let $J \in A(n)$. Let M be a subcontinuum in F $(b \in M)$, and C an (n - 1)-sphere. Denote by $(C \times M, b)$ the decomposition space [9, pp. 273-274] of the upper-semicontinuous decomposition of the cartesian product

$C \times M$, where the only nondegenerate element is taken as $C \times b$ (intuitively, we regard the decomposition space as a sort of generalized cone with vertex at the point $C \times b$). With this notation, we give the following definition.

DEFINITION 1. We say that F is an $A(n)$-*cartesian membrane from* b *to* J if and only if there exists a homeomorphism h from $(C \times M, b)$ onto F, for some M such that

(i) for some $a \in M - b$, $J = h(C \times a)$,

(ii) for all $q \in (M - b)$, $h(C \times q) \in A(n)$, and

(iii) $h(C \times b) = b$.

For brevity we occasionally say that F is an $A(n)$-*cartesian membrane with respect to* J. If moreover M is irreducible from b to a, then we call the membrane *irreducible.*

THEOREM 2. *Let* S *be a perfectly separable, compact Hausdorff space, and let* $n > 1$. *Then* S *is an* n-*sphere if and only if*

(1) *the class* $A(n)$ *of* $(n - 1)$-*spheres in* S *is not void;*

(2) *for each* $J \in A(n)$, $S = F_1 + F_2$, *where* F_1 *and* F_2 *are irreducible* $A(n)$-*cartesian membranes with respect to* J *such that* $F_1 \not\supset F_2$ *and* $F_2 \not\supset F_1$; *and whenever* S *is such a union and* F_3 *is any other* $A(n)$-*cartesian membrane containing* J, *then* F_3 *contains either* F_1 *or* F_2, *but not both; and*

(3) *if* $J \in A(n)$ *and* $p \in (S - J)$, *then there exists an* $A(n)$-*cartesian membrane from* p *to* J.

Proof of the Sufficiency. Let $J \in A(n)$; then by (2) of the hypothesis, $S = F_1 + F_2$, where F_1 and F_2 are irreducible $A(n)$-cartesian membranes with respect to J, and where $F_1 \not\supset F_2$ and $F_2 \not\supset F_1$. Now suppose that F_3 is also an irreducible $A(n)$-cartesian membrane with respect to J. By (2), F_3 contains F_1 say, but $F_3 \not\supset F_2$. Then $S = F_3 + F_2$, where $F_3 \not\supset F_2$ and $F_2 \not\supset F_3$. Hence again, by (2), F_1 contains either F_3 or F_2; since $F_1 \not\supset F_2$, $F_1 \supset F_3$. Therefore $F_3 = F_1$. We have proved that an irreducible $A(n)$-cartesian membrane with respect to J equals either F_1 or F_2. Let us designate this result by (R_1).

It follows easily from (2) and (R_1) that if F is an irreducible $A(n)$-cartesian membrane with respect to $J \in A(n)$, then no proper $A(n)$-cartesian membrane of F contains J. We designate this result by (R_2).

Now, above, let $F_1 = h_1(C \times M_1, b_1)$, where h_1 is a homeomorphism, as in Definition 1. Then M_1 is a continuum irreducible, say, from a to b_1, where $h_1(C \times a) = J$ because of (i) in Definition 1. We now wish to prove that Kuratowski's $I(b_1, M_1)$ reduces to a. Since $a \in I(b_1, M_1)$ by definition, suppose that $d \in I(b_1, M_1) - a$. By the definition, F_1 is an irreducible $A(n)$-cartesian membrane with respect to $J_0 = h_1(C \times d)$.

Let $q \in J_0$. By (3) of the hypothesis, there exists an $A(n)$-cartesian membrane F'_3 from q to J. Say $F'_3 = h_3(C \times M'_3, q)$, where h_3 is a homeomorphism, $h_3(C \times q) = q$, and for some $r \in M'_3$, $h_3(C \times r) = J$ above. Let M_3 be a continuum in M'_3 irreducible from q to r. Let $F_3 = h_3(C \times M_3, q)$; then F_3 is an irreducible $A(n)$-cartesian membrane with respect to J, and therefore, $F_3 = F_1$ or $F_3 = F_2$ by (R_1).

Let us consider the case $F_3 = F_1$.

Define a function $f\colon (C\times M_3, q) \to M_3$ by saying $f(c, x) = x$ for all $c \in C$ and $x \in M_3$. Then f is continuous and closed. The $(n - 1)$-sphere $h_3^{-1}(J_0)$ contains the point $h_3^{-1}(q)$ of $(C\times M_3, q)$ because $q \in J_0$; and $h_3^{-1}(J_0) \cap (C\times r) = \emptyset$ because $h_3(C\times r) = J$ and $J_0 \cap J = \emptyset$. Thus $N = f(h_3^{-1}(J_0))$ is a continuum in M_3 containing q but not r. Hence the set

$$h_3(C\times N, q) \subset h_3(C\times M_3, q) = F_3 = F_1$$

is a proper $A(n)$-cartesian membrane of F_1 that contains J_0; this contradicts (R_2), and so $F_3 \neq F_1$.

Consider now the case $F_3 = F_2$.

Here we may assume that $J_0 \subset F_2$; for otherwise we could have chosen q above so that $q \notin F_2$, thus eliminating the possibility that $F_3 = F_2$. We know by (2) of the hypothesis that $S = G_1 + G_2$, where G_1 and G_2 are irreducible $A(n)$-cartesian membranes with respect to J_0 such that $G_1 \not\supset G_2$ and $G_2 \not\supset G_1$. By (R_1), F_1 must equal one of these, say $F_1 = G_1$. By the method used in the case $F_3 = F_1$, we can construct an $A(n)$-cartesian membrane H in F_2 that contains J_0 but does not intersect J. By (2) of the hypothesis, H must contain either G_1 or G_2. But $H \not\supset J$ and $J \subset F_1 = G_1$; therefore $H \supset G_2$, and thus $G_2 \cap J = \emptyset$.

Now let $q' \in J$, and let G_3 be an irreducible $A(n)$-cartesian membrane from q' to J_0. Then either $G_3 = G_1$ or $G_3 = G_2$; since $q' \notin G_2$, $G_3 = G_1$. Now $G_3 = G_1$ is irreducible from q' to J_0, where $q' \in J$. Again we can revert to the method used in the case $F_3 = F_1$ to say that there exists a proper $A(n)$-cartesian membrane of $G_1 = F_1$, that contains J but not J_0: this contradicts (R_2), and so $F_3 \neq F_2$.

Thus the proof is complete for $I(b_1, M_1) = a$. Let us call this result (R_3).

We now show that $F_1 \cap F_2 = J$.

Since $F_2 \not\supset F_1$, let $z \in (F_1 - F_2)$. By (3) of the hypothesis there exists an $A(n)$-cartesian membrane F'_1 from z to J. As previously noted, we may choose F'_1 irreducible, which we do. By (R_1), and since $z \notin F_2$, it follows that F'_1 coincides with F_1, which contains J. Hence we may as well assume that F_1 was chosen as F'_1; that is, that $F_1 = h_1(C\times M_1, z)$, where $h_1(C\times z) = z$.

Now suppose $r \in ((F_1 \cap F_2) - J)$; then $r \neq z$. Let $h_1^{-1}(r) = (c', r')$ $(c' \in C$ and $r' \in M_1)$, and denote by J_r the set $h_1(C\times r')$. Then $J_r \in A(n)$ by (ii) of Definition 1. By (R_3) and properties of irreducible continua, there exists a proper subcontinuum N_1 of M_1 irreducible from z to r; that is, $a \notin N_1$. The irreducible $A(n)$-cartesian membrane $H_1 = h_1(C\times N_1, z)$ is a proper subset of $h_1(C\times M_1, z) = F_1$; moreover, since $a \notin N_1$, $J \cap H_1 = \emptyset$. Also, H_1 is irreducible from z to J_r because $h_1(C\times r') = J_r$.

By (2) of the hypothesis and (R_1), there exists an irreducible $A(n)$-cartesian membrane H_2 from a point p to J_r such that $H_1 \not\supset H_2$, $H_2 \not\supset H_1$, and $S = H_1 + H_2$. Since $H_1 \cap J = \emptyset$, $H_2 \supset J$. Let us say that $H_2 = g_2(C\times N_2, p)$, where g_2 is a homeomorphism and N_2 is a continuum irreducible from p to p'. Then $I(p, N_2) = p'$ by (R_3), and so $g_2(C\times p') = J_r$ because H_2 is irreducible.

Define a function $f'\colon (C\times N_2, p) \to N_2$ by saying $f'(c, x) = x$ for $c \in C$ and $x \in N_2$. Then f' is continuous and closed. Then $J \cap J_r = \emptyset$, the set $K = f'(g_2^{-1}(J))$ does not contain $p' = f'(g_2^{-1}(J_r))$, and K is a continuum since J is a continuum.

For the continuum N_2, $T(p') = \mathrm{cl}\,(I(p, N_2))$ by Theorem 12 of [3, p. 272]; hence $T(p') = p'$ by (R_3). Now let $k \in K$; then $p' \neq k$. By Theorem 2 of [4, p. 116], N_2 is symmetric with respect to the set-function T as defined in [4, p. 115]; hence $k \notin T(p')$ implies $p' \notin T(k)$. There are two possibilities. If $p \in T(k)$, let $I = K + T(k)$. If $p \notin T(k)$, then by Corollary 16.1 of [3, p. 274], T(k) separates p from p' in N_2. Then $N_2 - T(k)$ is the union of mutually separated sets K_1 and K_2, where $p \in K_2$, say. In this case define $I = T(k) + K_2 + K$. Since by (A), T(k) is a continuum, it follows from properties of continua that I is a continuum in either case.

The set $H' = g_2(C \times I, p)$ is an A(n)-cartesian membrane containing J but not intersecting J_r; thus $r \notin H'$, and so by (2) either $H' \supset F_1$ or $H' \supset F_2$, which is impossible since $r \in F_1 \cap F_2$ but $r \notin H'$.

This contradiction proves that $F_1 \cap F_2 = J$.

Thus J separates S. In general, every $J \in A(n)$ separates S. Recall that $F_1 = h_1(C \times M_1, b_1)$ and F_1 is an irreducible A(n)-cartesian membrane; we now prove that M_1 is an arc. Choose any point $y \in M_1 - a - b_1$ and consider $J_y = h_1(C \times y)$; $J_y \in A(n)$ by (ii) in Definition 1. By the above, J_y separates S; but J_y does not intersect F_2, and so J_y must separate the continuum F_1. By the construction of F_1, this implies that y separates M_1; therefore M_1 is irreducibly connected from a to b_1, and therefore it is an arc.

Thus F_1 is topologically equivalent to $(C \times [0, 1], 1)$, and therefore by Lemma 1 below, F_1, and similarly F_2, is a closed n-cell with boundary J. Since F_1 and F_2 meet only at their boundary J, the union $F_1 + F_2$ is an n-sphere. This completes the proof of the sufficiency.

LEMMA 1. *If* $F = (C \times [0, 1], 1)$ *and if* C *is an* (n - 1)-*sphere, then* F *is a closed* n-*cell.*

We do not prove this well-known result.

For the proof of the necessity in Theorem 2 we also need the following lemma.

LEMMA 2. *Let* S *be an* n-*sphere. If* C *is an* (n - 1)-*sphere,* M *is a nondegenerate continuum,* $p \in M$, *and* g *is a homeomorphism that maps* $(C \times M, p)$ *into* S, *then* M *is an arc and hence* $(C \times M, p)$ *is a closed* n-*cell.*

Proof. Let y be a non-cut point of M - p, and let N be an irreducible subcontinuum of M from p to y. By the Jordan-Brouwer Separation Theorem [11, p. 63], $g(C \times q)$ separates S into two complementary domains for $q \in M - p$. Let $G = \{g(C \times z): z \in N\}$, which is a decomposition of $g(C \times N, p)$. The hypothesis of Theorem 2 of [12, p. 1147] is satisfied, and from its conclusion the decomposition hyperspace $((C \times N, p), G)$ is locally connected and G is upper-semicontinuous. It follows that N is locally connected, since N is homeomorphic to $((C \times N, p), G)$. But N is an irreducible continuum and must be an arc. By Lemma 1, $g(C \times N, p)$ is a closed n-cell with boundary $g(C \times y)$, and therefore it must be one of the complementary domains of $g(C \times y)$. Since y is a non-cut point of M, all of $g(C \times M, p)$ must be contained in $g(C \times N, p)$, and so N = M is an arc.

DEFINITION 2. An (n - 1)-sphere C in an n-sphere S^n is said to be *flat* if and only if the closure of each complement of $S^n - C$ is a closed n-cell.

Proof of Necessity in Theorem 2.

We assume that S is an n-sphere $(n > 1)$. Let A(n) be the collection of flat (n - 1)-spheres in S. Since S is homeomorphic to the unit n-sphere, the class A(n) is nonnull. Let $J \in A(n)$. By the definition of flatness, $S - J = H' + K'$, where $cl(H') = H$ and $cl(K') = K$ are closed n-cells. By Lemma 1, $(C \times [0, 1], 1)$ is a closed n-cell when C is an (n - 1)-sphere; hence there exists a homeomorphism h from $(C \times [0, 1], 1)$ onto H. It follows from Brouwer's Invariance of Domain Theorem [6, p. 95] that $h(C \times 0) = J$, and therefore h satisfies (i) of Definition 1. Condition (ii) of Definition 1, is satisfied by the Generalized Schoenflies Theorem proved by Brown in [2, p. 76]. Now we merely note that $h(C \times 1)$ is mapped into a single point in H, and (iii) in the definition of an A(n)-cartesian membrane is satisfied; hence both H and K are A(n)-cartesian membranes.

Consider (2) of the theorem: let $S = H + K$, where H and K are A(n)-cartesian membranes satisfying (2). Let G be any other A(n)-cartesian membrane containing J. By Lemma 2, G is a closed n-cell; thus G must contain either H or K. Hence (2) of Theorem 2 is satisfied.

Part (3) of the theorem follows easily from our choice of A(n). This completes the proof of Theorem 2.

3. CONCLUDING REMARKS AND A LEMMA

In the proof of the sufficiency in Theorem 2, we used the condition

$$F_1 = h_1(C \times M_1, b_1),$$

where $h(C \times q) = J_q$ is an (n - 1)-sphere for $q \in M_1 - b_1$; however, without using this property of J_q we proved that M_1 must be an arc. But if J_q were an indecomposable continuum, for example (that is, if A(n) were a class of such continua), then condition (3) in Theorem 2 would be meaningless as a hypothesis. This suggests the question of whether A(n) could be replaced with a class of continua of another type, with (1), (2) and (3) holding; and if so, what kind of continuum S would have to be.

The referee informs us that Bing and Rosen have given examples of suspension spheres that are not topological spheres and that may be used for our class A(n).

The proof we gave for Theorem 1 partly parallels that of the sufficiency for Theorem 2. Instead, we could have made use of a different method that would also prove Lemma 3 below. We say that a continuum M is the *essential union* of subcontinua M_i ($i = 1, 2, \cdots, n$; n finite), if M is the union of these M_i and each M_i contains points not in the union of the remaining M_j ($j = 1, 2, \cdots, n$; $i \neq j$).

LEMMA 3. *Let* S *be a compact Hausdorff continuum, let* $\{P_\alpha\}$ *be the class of all pairs of disjoint points of* S, *and for each* α *let* $\{M_\beta\}_\alpha$ *be the class of all irreducible subcontinua of* S *joining* p *and* q *of* P_α. *If for each* α *and for each* $M_1 \in \{M_\beta\}_\alpha$ *there exist a finite* n_α *and sets* $M_j \in \{M_\beta\}_\alpha$ ($j = 1, 2, \cdots, n_\alpha$) *such that* S *is the essential union of these* n_α *sets, then* S *is locally connected.*

The proof follows quickly from Theorem 2.1 of [11, p. 102].

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