NON-FLAT EMBEDDINGS OF S^{n-1} IN S^n

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1. We consider certain embeddings of the (n - 1)-sphere S^{n-1} in the n-sphere S^n and will assume in every case that $n > 3$. It will be assumed that the reader is familiar with the proofs of Lemma 1 of [4] and Theorems 4 and 5 of [5].

DEFINITION 1. *An* (n - 1)-*sphere* S *in* S^n *is* flat *if the closure of each component of* S^n - S *is a closed* n-*cell.*

DEFINITION 2. *Let* D *be a* k-*cell in* S^n, *and let* p *be a point of* S^n - D. *We take coordinates* $x_1, x_2, \cdots, x_{n-1}, x_n$ *in* $E^n = S^n - p$ *and say that* D *is* flat *in* $S^n(E^n)$ *if there is a homeomorphism* h *of* E^n *onto itself such that* h(D) *is a unit cell in the hyperplane* $x_n = x_{n-1} = \cdots = x_{k+1} = 0$.

DEFINITION 3. *Let* S *be an* (n - 1)-*sphere in* S^n, *and let* G *be a component of* S^n - S. *We say that* S *has a local collar in* Cl G *at* $p \in S$ *if there exists a neighborhood* U *of* p, *relative to* S, *and a homeomorphism* h, *carrying* $U \times [0, 1]$ *into* Cl G, *such that* $h(x, 0) = x$ *for each* $x \in U$. *We say that* S *is locally flat at* p *if there exists a homeomorphism* f, *carrying* $U \times [-1, 1]$ *into* S^n, *such that* $f(x, 0) = x$ *for each* $x \in U$. *A similiar definition is given for a locally flat* (n - 1)-*cell in* S^n.

CONJECTURE. *If* $D = D_1 \cup D_2$, *where* D_1 *and* D_2 *are flat* (n - 1)-*cells in* $S^n(E^n)$ *and* $D_1 \cap D_2 = \mathrm{Bd}\,D_1 \cap \mathrm{Bd}\,D_2$ *is a flat* (n - 2)-*cell, then* D *is flat in* $S^n(E^n)$.

An (n - 1)-sphere in S^n is flat if and only if it is locally flat at each of its points [3]. Thus, if S is a non-flat (n - 1)-sphere in S^n, then there are points at which it fails to be locally flat, and we denote the set of all such points by E. If S has been constructed by one of the standard techniques (the horned sphere construction [2], spinning an (n - 2)-cell, suspending an (n - 2)-sphere, or capping a cylinder over an (n - 2)-sphere [5]), then E is an uncountable set. In this paper we show that E cannot consist of a single point (as can happen for n = 3 [1]). Furthermore it is shown that, if the above conjecture is true, E has no isolated points. Then, since E is closed, E will have to contain uncountably many points.

2. For each t ($0 \le t \le 1$) let A_t be the solid ball in E^n centered at the origin and with radius t. For $-1 \le t \le 1$, let

$$B_t = \{(x_1, \cdots, x_n) \mid x_1^2 + x_2^2 + \cdots + x_{n-1}^2 + (x_n + t)^2 \le (1 + t)^2\}.$$

We observe that the proof of Lemma 1 of [4] may be applied directly to establish the following lemma.

LEMMA 1. *Let* S *be an* (n - 1)-*sphere in* S^n, p *a point of* S, *and* G *a component of* S^n - S. *Suppose that* S *has a local collar in* Cl G *at each point of* S - p *and that* h *is a homeomorphism of* Bd A_1 *onto* S *such that* $h[(0, 0, \cdots, 0, 1)] = p$. *Then* h *can be extended to a homeomorphism of* Cl $(B_1 - A_1)$ *into* Cl G.

Lemma 1 and Theorem 1 of [3] imply the following lemma.

LEMMA 2. *Let* S, p, *and* h *be as in Lemma* 1, *and denote the components of* S^n - S *by* G, H. *Suppose that* S *is locally flat at each point of* S - p *and that* S *has*

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a local collar in Cl H *at* p. *Then* h *can be extended to a homeomorphism of* Cl $(B_1 - A_{1/2})$ *into* S^n.

THEOREM 1. *If* S *is as in Lemma* 2, *then* S *is flat.*

Proof. Let h be a homeomorphism of Cl $(B_1 - A_{1/2})$ into S^n such that $h(\text{Bd } A_1) = S$, $h[(0, 0, \cdots, 0, 1)] = p$, and $h(\text{Bd } A_{1/2}) \subset H$. By Theorem 1 of [5], we know that Cl H is a closed n-cell, and we proceed to show that Cl G is a closed n-cell.

Let L denote the closed segment of the x_n-axis from Bd $A_{2/3}$ to Bd A_1, and assume a combinatorial triangulation of Int $A_1 - A_{1/2}$ in which Bd $A_{2/3}$, Bd $A_{3/4}$, and L - (0, 0, ⋯, 0, 1) are polyhedra has been made. We assign to

$$h(A_{3/4} - \text{Int } A_{2/3})$$

the triangulation determined by h; and, since each boundary sphere of this annulus is flat, this triangulation can be extended to a combinatorial triangulation of S^n. For the remainder of this proof S^n will denote the sphere together with the above triangulation.

Let K be the closure of the component of $S^n - h(\text{Bd } A_{2/3})$ which contains S, and notice that K is a combinatorial n-cell in S^n. It was shown in the proof of Theorem 4 of [5] that, if there exists a continuous mapping k of K onto itself such that h(L) is the only inverse set under k (h(L) can be contracted to a boundary point of K), then Cl G is topologically equivalent to K. It was further shown that, if h(L) is locally polyhedral at each point different from p, then such a mapping k can be constructed. Thus it suffices to construct a homeomorphism f of K onto itself such that fh(L) is locally polyhedral at each point different from f(p).

Let M_1 be the set h(Int A_1 - Int $A_{2/3}$) together with the triangulation determined by h, and for each positive integer i let

$$t_i = h(L \cap \text{Bd } A_{\frac{i+1}{i+2}}).$$

Then, if L_1 is the closed subarc of h(L) from t_1 to t_2 and P_i (i = 1, 2, ⋯) is the closed subarc of h(L) from t_1 to t_{i+2}, we see that L_1 is polyhedral in both S^n and M_1, and that P_i is polyhedral in M_1. Let $\varepsilon_1 > 0$ be so small that U_1 (the closure of the ε_1-neighborhood of $P_1 - L_1$) does not meet Bd K, Bd G, or $h(L) - P_2$. Then apply Theorem 2.1 of [6] to obtain an ε_1-homeomorphism f_1 of K onto itself and such that: (1) f_1 is the identity outside U_1 and on L_1 and (2) f_1 is semi-linear on P_1.

Suppose that $i > 1$ and that certain homeomorphisms $f_{i-1}, \cdots, f_2, f_1$ of K onto K have been constructed so that, if M_i is the set $f_{i-1} \cdots f_2 f_1 (M_1)$ together with the triangulation determined by M_1 and $f_{i-1} \cdots f_2 f_1$, then $L_i = f_{i-1} \cdots f_2 f_1 (P_{i-1})$ is polyhedral in both S^n and M_i. If i = 2, let $\varepsilon_2 > 0$ be so small that U_2 (the closure of the ε_2-neighborhood of $f_1(P_2 - L_2)$) does not meet Bd K, Bd G, or

$$f_1 h(L) - f_1(P_3).$$

For $i > 2$ let $\varepsilon_i > 0$ be so small that U_i (the closure of the ε_i-neighborhood of $f_{i-1} \cdots f_1(P_i) - L_i$) does not meet Bd K,

$$\text{Bd } G,\ f_{i-1} \cdots f_1 h(L) - f_{i-1} \cdots f_1(P_{i+1}),$$

$\bigcup_{j=1}^{i-2} U_j$, or $\bigcup_{j=1}^{i-2} f_{i-1} \cdots f_1(U_j)$. Then apply Theorem 2.1 of [6] to obtain an ε_i-homeomorphism f_i of K onto K such that: (1) f_i is the identity outside U_i and on L_i and (2) f_i is semi-linear on $f_{i-1} \cdots f_1(P_i)$.

The homeomorphism f of K onto K is then defined by $f(x) = \lim_{i\to\infty} f_i \cdots f_1(x)$. Routine verifications show that f has the desired properties, and the proof of Theorem 1 is complete.

The effect of Theorem 1 is to remove the semi-linear condition in Theorem 4 of [5]. As was observed in [5], this allows one to remove the semi-linear condition in Theorem 5 of [5]. Thus, with Lemma 1, we have the following theorem.

THEOREM 2. *If* S *and* G *are as in Lemma* 1, *then* Cl G *is a closed* n-*cell.*

COROLLARY. *If* S *is an* (n - 1)-*sphere in* S^n, $p \in S$, *and* S *is locally flat at each point of* S - p, *then* S *is flat in* S^n.

Proof. If we denote the components of $S^n - S$ by G and H, then Theorem 2 implies that both Cl G and Cl H are closed n-cells.

3. In considering the question of existence of isolated points of E, we would like to consider a point $p \in S$ for which there exists a neighborhood U of p (relative to S) such that S is locally flat at each point of U - p and to show that S is locally flat at p. To do this it would suffice to show that for an (n - 1)-cell K in S^n and $p \in \text{Int } K$, K is locally flat at p if K is locally flat at each point of K - p.

Let K be as described above, E_0^n the half space of E^n defined by $x_n \geq 0$, and h a homeomorphism of $\text{Bd } A_1 \cap E_0^n$ onto K such that $h[(0, 0, \cdots, 0, 1)] = p$. By a procedure entirely analogous to a proof of Lemma 1, we can establish the following lemma.

LEMMA 3. *There is a homeomorphic extension of* h *which carries* $(B_1 - \text{Int } B_{-1/3}) \cap E_0^n$ *into* S^n.

LEMMA 4. *Let* S *be a flat* (n - 1)-*sphere in* S^n, *and let* L *be either an* (n - 1)-*cell in* S *or an* (n - 2)-*cell in* S. *If* L *is flat in* S, *then* L *is flat in* S^n.

THEOREM 3. *Let* K *be an* (n - 1)-*cell in* S^n, *and let* p *be an interior point of* K. *If* K *is locally flat at each point of* k - p, *then, if the conjecture is true,* K *is locally flat at* p.

Proof. Let h be the homeomorphism given by Lemma 3. Let $E_{1/2}^n$ be the half space of E^n defined by $x_n \geq 1/2$, and consider the sets M_1, M_2, S_1, S_2, D_1, D_2, where M_1 is the n-cell consisting of the part of $(B_{1/2} - \text{Int } A_1) \cap E_{1/2}^n$ determined by $x_{n-1} \geq 0$, M_2 is the cell consisting of the part of $(B_{1/2} - \text{Int } A_1) \cap E_{1/2}^n$ determined by $x_{n-1} \leq 0$, $S_1 = \text{Bd } M_1$, $S_2 = \text{Bd } M_2$, $D_1 = S_1 \cap \text{Bd } A_1$, and $D_2 = S_2 \cap \text{Bd } A_1$.

We let f be the restriction of h to $S_1 \cup S_2$, and observe that, for i = 1, 2, f can be extended into the interior of S_i at each point and into the exterior at each point different from (0, 0, ⋯, 0, 1). Thus, by Theorem 1, $f(S_1)$ and $f(S_2)$ are flat in S^n. Furthermore, it is clear that $f(D_i)$ is flat in $f(S_i)$, i = 1, 2, and that $f(D_1) \cap f(D_2)$ is flat in both $f(S_1)$ and $f(S_2)$. Then, by the conjecture,

$$f(D_1) \cup f(D_2) = f(D_1 \cup D_2) = f(\text{Bd } A_1 \cap E_{1/2}^n)$$

is flat in S^n and must be locally flat at p. Hence K is locally flat at p.

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