AN INTEGRAL INEQUALITY WITH APPLICATIONS TO HARMONIC MAPPINGS

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1. INTRODUCTION

This paper concerns a type of extremal problem in the theory of harmonic mappings first posed by E. Heinz [3]. Heinz considered a class of harmonic mappings H, and showed that a particular measure of local distortion evaluated at the origin has a positive greatest lower bound ρ. The value of ρ has not yet been determined, the latest estimate being in [1]. We first establish an inequality for the absolute value of a class of integrals with complex-valued integrand, and then apply it to a problem of this nature. Let H_1 be the odd mappings of the class H. We consider a different, but related, measure of local distortion applied at the origin and obtain the value of the greatest lower bound for the class H_1.

The integral inequality is established in Section 2, and is applied to harmonic mappings in Section 3. Section 4 contains detailed proofs of some needed lemmas which are also presented in [2], but more sketchily.

2. THE INTEGRAL INEQUALITY

THEOREM 1. *Let* $h(\theta)$ *be a non-decreasing continuous function on* $[0, \pi]$ *such that* $h(0) = 0$ *and* $h(\pi) = \pi$. *Then*

$$(2.1)\qquad \left| \int_0^\pi \exp(i(h(\theta) - \theta))\, d\theta \right| > 2,$$

and 2 *cannot be replaced by any greater number.*

Proof. We divide the proof into five sections, (a) to (e), which we first outline. In (a) we state six lemmas that will be needed at various stages. The proofs of the first two are put in the appendix, and the others are proved following their statement. These lemmas are all concerned with the properties of the rearrangement of a function. (b) By Lemmas 1, 2 and 3 there is a non-decreasing, continuous function on $[0, \pi]$, say $g^*(\theta)$ which is equimeasurable with $h(\theta) - \theta$, and therefore satisfies

$$(2.2)\qquad \int_0^\pi \exp(i(h(\theta) - \theta))\, d\theta = \int_0^\pi \exp(ig^*(\theta))\, d\theta .$$

(c) We next make use of the inequality established in Lemma 4, namely

$$g^*(\theta'') - g^*(\theta') \leq \theta'' - \theta',$$

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where $0 \leq \theta' < \theta'' \leq \pi$, to prove that

$$\left| \int_0^{\pi} \exp(ig^*(\theta))\, d\theta \right| \geq 2 . \tag{2.3}$$

(d) Making use of Lemma 6, we show that equality cannot hold in (2.3). (e) Finally, for any $\varepsilon > 0$, we demonstrate that an $h(\theta)$ exists such that the left side of (2.1) does not exceed $2 + \varepsilon$. This completes the demonstration.

(a) LEMMA 1. *Let* $f(\theta)$ *be a measurable function on* $[0, \pi]$ *with essential lower bound* m, *and essential upper bound* M, $-\infty < m < M < \infty$. *There exists a non-decreasing function* $f^*(\theta)$ *on* $[0, \pi]$ *which has the properties:*

$$f^*(0) = m,\quad f^*(\pi) = M, \tag{2.4}$$

$$\text{meas } E(f(\theta) \leq \gamma) = \text{meas } E(f^*(\theta) \leq \gamma) . \tag{2.5}$$

Remarks. The notation E( ), where the parentheses contain an equality or inequality, refers to the set of points where the equality or inequality holds. The function $f^*(\theta)$ of Lemma 1 is called a non-decreasing rearrangement of $f(\theta)$.

We frequently need

$$\text{meas } E(\alpha < f(\theta) < \beta) = \text{meas } E(\alpha < f^*(\theta) < \beta) , \tag{2.6}$$

which follows from (2.5).

Proof. See [2, p. 276] and Section 4.

LEMMA 2. *Let* $f(\theta)$ *be a measurable function on* $[0, \pi]$ *with essential lower bound* m, *and essential upper bound* M, $-\infty < m < M < \infty$. *Let* $f^*(\theta)$ *be its non-decreasing rearrangement and let* $F(x)$ *be any continuous function on* $[m, M]$. *Then*

$$\int_0^{\pi} F(f(\theta))\, d\theta = \int_0^{\pi} F(f^*(\theta))\, d\theta . \tag{2.7}$$

Proof. See [2, p. 277] and Section 4.

LEMMA 3. *Let* $f(\theta)$ *be a continuous, non-constant function on* $[0, \pi]$. *The non-decreasing rearrangement* $f^*(\theta)$ *of* $f(\theta)$ *is a continuous function.*

Proof. Since $f(\theta)$ is not a constant, then $m < M$ (see Lemma 1, hypothesis). Since $f^*(0) = m$, $f^*(\pi) = M$ and $f^*(\theta)$ is non-decreasing, if it is discontinuous, then it must omit values in the interval $[\alpha, \beta]$, where $m \leq \alpha < \beta \leq M$. Thus

$$\text{meas } E(\alpha < f^*(\theta) < \beta) = 0 .$$

By (2.6) this implies that

$$\text{meas } E(\alpha < f(\theta) < \beta) = 0 . \tag{2.8}$$

When we show that the continuity of $f(\theta)$ implies that (2.8) cannot hold, the proof of the lemma will be complete.

To establish that (2.8) cannot hold, we first note that $E_1 = E(f(\theta) = \alpha)$ and $E_2 = E(f(\theta) = \beta)$ are non-empty, disjoint, closed sets. Let θ_1 be a point of E_1 and let θ_2 be the closest point of E_2 to θ_1. Say it is to the right of θ_1. In the other case the adjustment to the following argument will be obvious. Let θ_1' be the closest point of E_1, that also lies in $[\theta_1, \theta_2]$, to the point θ_2. By this construction there are no points of E_1 or E_2 on (θ_1', θ_2). Since $f(\theta_1') = \alpha$, $f(\theta_2) = \beta$, it must follow that

$$\alpha < f(\theta) < \beta$$

on (θ_1', θ_2), and (2.8) is contradicted.

LEMMA 4. *Let* $h(\theta)$ *be a non-decreasing, continuous function on* $[0, \pi]$ *and let* $h(0) = 0$, $h(\pi) = \pi$. *Let* $g(\theta) = h(\theta) - \theta$, *and let* $g^*(\theta)$ *be the non-decreasing re-arrangement of* $g(\theta)$. *Then*

$$g^*(\theta'') - g^*(\theta') \leq \theta'' - \theta', \tag{2.9}$$

where $0 \leq \theta' < \theta'' \leq \pi$.

Proof. (1) We can assume that $g^*(\theta') \neq g^*(\theta'')$; otherwise, (2.9) is true. Since $g^*(\theta)$ is continuous, there is a least value of θ, say $\bar{\theta}''$, such that

$$g^*(\bar{\theta}'') = g^*(\theta''),$$

and a greatest value of θ, say $\bar{\theta}'$, such that

$$g^*(\bar{\theta}') = g^*(\theta').$$

These values satisfy the inequalities

$$\theta' \leq \bar{\theta}' < \bar{\theta}'' \leq \theta''.$$

If we can prove

$$g^*(\bar{\theta}'') - g^*(\bar{\theta}') \leq \bar{\theta}'' - \bar{\theta}', \tag{2.10}$$

then the truth of (2.9) will be established.

(2) Since

$$g^*(\bar{\theta}') < g^*(\theta) < g^*(\bar{\theta}'')$$

for $\bar{\theta}' < \theta < \bar{\theta}''$,

$$\text{meas } E(g^*(\bar{\theta}') < g^*(\theta) < g^*(\bar{\theta}'')) = \bar{\theta}'' - \bar{\theta}'.$$

By (2.6) it follows that

$$\text{meas } E(g^*(\bar{\theta}') < g(\theta) < g^*(\bar{\theta}'')) = \bar{\theta}'' - \bar{\theta}'. \tag{2.11}$$

(3) We next want to show that

$$\text{meas } E(g^*(\bar{\theta}') < g(\theta) < g^*(\bar{\theta}'')) \geq g^*(\bar{\theta}'') - g^*(\bar{\theta}').$$

Combining this inequality with (2.11) then yields (2.10), completing the proof of the lemma. Let $m = \min g(\theta)$, $M = \max g(\theta)$, $0 \leq \theta \leq \pi$, and let $\alpha = g^*(\bar{\theta}')$, $\beta = g^*(\bar{\theta}'')$. Then $m \leq \alpha < \beta \leq M$. In this notation we want to prove

$$\text{(2.12)} \qquad \text{meas } E(\alpha < g(\theta) < \beta) \geq \beta - \alpha .$$

(4) We define a function $h_1(\theta)$ on $[0, 2\pi]$ as follows. Let $h_1(\theta) = h(\theta)$ on $[0, \pi]$ and let $h_1(\theta) = h(\theta - \pi) + \pi$ on $[\pi, 2\pi]$. Since $h(0) = 0$, we see that $h_1(\theta)$ is continuous on $[0, 2\pi]$, and we also note that it is non-decreasing on $[0, 2\pi]$.

Let $g_1(\theta) = h_1(\theta) - \theta$. We will use the relationship

$$\text{(2.13)} \qquad g_1(\theta + \pi) = g_1(\theta) \quad (0 \leq \theta \leq \pi) .$$

By (2.13) we find that

$$\text{(2.14)} \qquad \text{meas } E(\alpha < g_1(\theta) < \beta) \quad (\alpha \leq \theta \leq \alpha + \pi) ,$$

where $0 \leq \alpha \leq \pi$, is the same as

$$\text{(2.15)} \qquad \text{meas } E(\alpha < g_1(\theta) < \beta) \quad (0 \leq \theta \leq \pi) ,$$

which in turn equals

$$\text{(2.16)} \qquad \text{meas } E(\alpha < g(\theta) < \beta) .$$

Remarks on the notation of (2.14) *and* (2.15). When θ has the added restriction of belonging to a smaller set than the domain of definition of the function involved in the $E(\ )$ notation, this extra restriction is added to the right.

(5) Let θ_1 be the smallest value of θ on $[0, \pi]$ at which $g_1(\theta)$ attains the value β. The value m will be obtained on the interval $[\theta_1, \theta_1 + \pi]$, so by the intermediate value property of continuous functions, the value α is obtained in this interval. Let θ_2 be the smallest value of θ on $[\theta_1, \theta_1 + \pi]$ where this happens. Finally, let θ_1' be the largest value of θ on the interval $[\theta_1, \theta_2]$ where $g_1(\theta)$ attains the value β. By this construction,

$$\alpha < g_1(\theta) < \beta \qquad (\theta_1' < \theta < \theta_2) ,$$

and so

$$\text{(2.17)} \qquad \text{meas } E(\alpha < g_1(\theta) < \beta) \geq \theta_2 - \theta_1' \quad (\theta_1 \leq \theta \leq \theta_1 + \pi) .$$

Also

$$\text{(2.18)} \qquad \beta - \alpha = g_1(\theta_1') - g_1(\theta_2) = h_1(\theta_1') - h_1(\theta_2) + (\theta_2 - \theta_1') \leq \theta_2 - \theta_1' .$$

Combining (2.15) and (2.16) with (2.17) and (2.18) yields (2.12), and so the lemma is demonstrated.

LEMMA 5. *Let* $g^*(\theta)$ *be as defined in Lemma* 4. *Then*

$$\text{(2.19)} \qquad g^*(\pi) - g^*(\theta) < \pi .$$

Proof. This follows as a special case of (2.18) when we chose $\beta = M$, $\alpha = m$, where M is the maximum and m the minimum of $g_1(\theta)$. The value $\theta_2 - \theta_1'$ which arises in this case must be less than π since m is achieved on the interval $[\theta_1, \theta_1 + \pi)$.

LEMMA 6. *Let* $g^*(\theta)$ *be as defined in Lemma* 4. *Then either*

$$g^*\left(\frac{\pi}{2}\right) - g^*(0) < \frac{\pi}{2} \tag{2.20}$$

or

$$g^*(\pi) - g^*\left(\frac{\pi}{2}\right) < \frac{\pi}{2}. \tag{2.21}$$

By Lemma 4 we know that (2.20) and (2.21) are true when $<$ is replaced by $\leq$. Hence if the conclusion is contradicted, equality holds in both (2.20) and (2.21). Addition of these equations, however, leads to a contradiction of (2.19).

(b) The equality (2.2) follows now from Lemmas 1, 2 and 3.

(c) We now consider the right-hand side of (2.2). First note that

$$\mathfrak{R} \exp\left(-ig^*\left(\frac{\pi}{2}\right)\right) \int_0^\pi \exp(ig^*(\theta))\, d\theta \leq \left| \int_0^\pi \exp(ig^*(\theta))\, d\theta \right|. \tag{2.22}$$

The left member of (2.22) is equal to

$$\begin{aligned} &\int_0^\pi \cos\left(g^*(\theta) - g^*\left(\frac{\pi}{2}\right)\right) d\theta \\ &= \int_0^{\frac{\pi}{2}} \cos\left(g^*\left(\frac{\pi}{2}\right) - g^*(\theta)\right) d\theta + \int_{\frac{\pi}{2}}^{\pi} \cos\left(g^*(\theta) - g^*\left(\frac{\pi}{2}\right)\right) d\theta. \end{aligned} \tag{2.23}$$

Lemma 4 implies the following inequalities

$$0 \leq g^*\left(\frac{\pi}{2}\right) - g^*(\theta) \leq \frac{\pi}{2} - \theta \leq \frac{\pi}{2} \quad \left(0 \leq \theta \leq \frac{\pi}{2}\right), \tag{2.24}$$

and

$$0 \leq g^*(\theta) - g^*\left(\frac{\pi}{2}\right) \leq \theta - \frac{\pi}{2} \leq \frac{\pi}{2} \quad \left(\frac{\pi}{2} \leq \theta \leq \pi\right). \tag{2.25}$$

Hence we find that the right member (2.23) is not less than

$$\int_0^{\frac{\pi}{2}} \cos\left(\frac{\pi}{2} - \theta\right) d\theta + \int_{\frac{\pi}{2}}^{\pi} \cos\left(\theta - \frac{\pi}{2}\right) d\theta = 2. \tag{2.26}$$

(d) By Lemma 6, we can actually deduce that (2.23) is greater than (2.26).

(e) Let ε be a positive number, and let

$$h_\delta(\theta) = \begin{cases} h_\delta^1(\theta) = \dfrac{\theta\pi}{2\delta}, & 0 \leq \theta \leq \delta, \\ \dfrac{\pi}{2}, & \delta \leq \theta \leq \pi - \delta, \\ h_\delta^2(\theta) = \dfrac{\pi}{2} + (\theta - (\pi - \delta))\dfrac{\pi}{2\delta}, & \pi - \delta \leq \theta \leq \pi, \end{cases}$$

where $0 < \delta < \pi/2$. Replace $h(\theta)$ by $h_\delta(\theta)$ in the left member of (2.1). Then

$$\left|\int_0^\pi \exp(i(h_\delta(\theta) - \theta))\, d\theta\right|$$

$$\leq \left|\int_0^\delta \exp i\,(h_\delta^1(\theta) - \theta)\, d\theta\right| + \left|\int_\delta^{\pi-\delta} \exp i\left(\frac{\pi}{2} - \theta\right) d\theta\right| \tag{2.27}$$

$$+ \left|\int_{\pi-\delta}^\pi \exp(i(h_\delta^2(\theta) - \theta))\, d\theta\right| \leq 2\delta + 2\sin\left(\frac{\pi}{2} - \delta\right).$$

For sufficiently small δ, the last term of (2.27) is less than $2 + \varepsilon$, and this completes the demonstration of Theorem 1.

3. APPLICATION TO HARMONIC MAPPINGS

Let H be the class of mappings

$$H(x, y) = u(x, y) + iv(x, y),$$

where

(a) $u(x, y)$ and $v(x, y)$ are harmonic functions for $x^2 + y^2 < 1$ and continuous for $x^2 + y^2 \leq 1$,

(b) $u(0, 0) = v(0, 0) = 0$,

(c) the values of $H(x, y)$ cover the circle $\{u^2 + v^2 \leq 1\}$ in a one-to-one, sense-preserving manner and

(d) $H(1) = 1$.

Let H_1 be the subclass of H for which the condition

(e) $H(-x, -y) = -H(x, y)$

is also satisfied.

It was shown by Heinz [3] that

$$M = u_x^2(0, 0) + u_y^2(0, 0) + v_x^2(0, 0) + v_y^2(0, 0)$$

has a positive greatest lower bound ρ for the class of mappings H, and also that $\rho \geq .358 \cdots$. This inequality is discussed and improved upon in [1] and [4].

Let

$$J = u_x(0, 0)\,v_y(0, 0) - u_y(0, 0)\,v_x(0, 0),$$

let λ be the greatest lower bound of $M + 2J$ for the class H, and let λ_1 be the greatest lower bound for the class H_1.

THEOREM 2. *The equality*

$$\lambda_1 = \frac{16}{\pi^2}$$

holds. Furthermore, this bound is not attained by $M + 2J$ *in the class* H_1.

Remarks. The quantity λ has an elementary geometric interpretation in terms of the distortion of $H(x, y)$ at the origin which supports its investigation. We conjecture that

$$\lambda = \frac{16}{\pi^2},$$

and that the bound is not attained by $M + 2J$ in the class H.

Proof. Let $H(x, y) = u(x, y) + iv(x, y)$ be an element of H, and let

$$w(z) = u(x, y) + iu^*(x, y) + i(v(x, y) + iv^*(x, y)), \tag{3.1}$$

where $u^*(x, y)$ and $v^*(x, y)$ are the harmonic conjugates of $u(x, y)$ and $v(x, y)$, respectively, chosen so that $w(0) = 0$. A computation yields the equality

$$|w'(0)|^2 = M + 2J. \tag{3.2}$$

The function $w(z)$ is analytic, and $w'(0)$ indicates its derivative at the origin.

We next seek an integral representation for $w(0)$. By the hypothesis

$$H(\cos\theta, \sin\theta) = \exp ih(\theta),$$

where $h(\theta)$ is a continuous, increasing function on $[0, 2\pi]$, and $h(0) = 0$, $h(2\pi) = 2\pi$. This follows because $H(x, y)$ performs a one-to-one, sense-preserving homeomorphism of $x^2 + y^2 = 1$ onto $\{u^2 + v^2 = 1\}$, taking $(1, 0)$ into $(1, 0)$. For the class H_1, to which we now restrict ourselves, the equality

$$h(\theta + \pi) = h(\theta) + \pi, \qquad 0 \leq \theta \leq \pi, \tag{3.3}$$

also holds.

The Poisson representation for harmonic functions yields the representation

$$H(x, y) = \frac{1}{2\pi} \int_0^{2\pi} \exp(ih(\theta))\, \Re \frac{\zeta + (x + iy)}{\zeta - (x + iy)}\, d\theta$$

where $\zeta = \exp i\theta$. From this we obtain the equality

$$w(z) = \frac{1}{2\pi} \int_0^{2\pi} \exp(ih(\theta)) \frac{\zeta + (x + iy)}{\zeta - (x + iy)} d\theta ,$$

and find that

$$w'(0) = \frac{1}{\pi} \int_0^{2\pi} \exp(i(h(\theta) - \theta))\, d\theta .$$

Because of (3.3) this can be written as

$$w'(0) = \frac{2}{\pi} \int_0^{\pi} \exp i((h(\theta) - \theta))\, d\theta .$$

The hypotheses for Theorem 1 are satisfied, and hence

$$|w'(0)| > \frac{4}{\pi}.$$

By (3.2), we then obtain the fact that $\lambda_1 \geq 16/\pi^2$. The following example shows that $\lambda_1 = 16/\pi^2$.

A specific example shows that the constant is the best. Let $r = s + it = f_c(z)$ be the analytic function which maps $|z| < 1$ onto the interior of the ellipse $(s/c)^2 + t^2 = 1$, taking zero into zero and having positive derivative at the origin. Let $h_c(r) = it + s/c$, where c is a positive constant. The mapping $h_c(f_c(z))$ is in class H_1 for all values of c.

Let $M_c + 2J_c$ be the value of $M + 2J$ for a fixed value of c. We will now show that $M_c + 2J_c$ has the limit $16/\pi^2$ as c tends to infinity. Let $w_c(z)$ be the function defined by (3.1). In our example,

$$w_c(z) = f_c(z) \left(\frac{1}{c} + i\right) .$$

Thus

$$M_c + 2J_c = |f_c'(0)|^2 \left|1 + \frac{1}{c^2}\right| . \tag{3.4}$$

As c tends to infinity, $f_c(z)$ will tend uniformly in $|z| \leq \rho < 1$ to the function which maps $|z| < 1$ onto the domain $-1 < \Im w < 1$, taking 0 into 0 with positive derivative at the origin. One way to see this is to observe that $-1 < \Im w < 1$ is the kernal of the sequence of ellipses $(\Re w/c)^2 + (\Im w)^2 < 1$, where c assumes a sequence of values tending to infinity [1, p. 76]. The quantity $|f_c'(0)|$ thus converges to the derivative of the limit function at $z = 0$, and a computation shows this to be $4/\pi$. This enables us to compute the limit of $M_c + 2J_c$ from (3.4) to complete the demonstration.

4. APPENDIX ON REARRANGEMENTS

Proof of Lemma 1. Let

$$P(y) = \text{meas } E(f(\theta) \leq y) \qquad (m \leq y \leq M). \tag{4.1}$$

This is called the measure function of $f(\theta)$. $P(y)$ is a non-decreasing function on $[m, M]$, since if $y' > y$

$$P(y') - P(y) = \text{meas } E(y < f(\theta) \leq y') \geq 0. \tag{4.2}$$

It also follows from (4.2) that $P(y)$ is continuous from the right. We observe at this point that

$$P(m) \geq 0$$

and that

$$P(M) = \pi. \tag{4.3}$$

The function $P(y)$ has a denumerable number of discontinuities, say at α_i, with an interval of values A_i omitted at each point. The intervals A_i have the form $[a_i, a_i')$ or (a_i, a_i'). The least upper bound of the interval is assumed because $P(y)$ is continuous from the right. Let R denote the range of $P(y)$. It is the subset of $[0, \pi]$ obtained by deleting the intervals A_i, and the interval $[0, \varepsilon)$ in the case $P(m) = \varepsilon > 0$.

Let $0 \leq \beta \leq \pi$. If $P(y) = \beta$ has more than one solution, the solutions form an interval B. The value β will be called a value of constancy if more than one solution exists, and B will be called an interval of constancy. $P(y)$ has at most a denumerable number of values of constancy, say β_i, with corresponding intervals B_i $(i = 1, 2, \cdots)$. The interval B_i has the form $[b_i, b_i')$ or $[b_i, b_i']$. These intervals are closed on the left because $P(y)$ is continuous from the right. We note that zero cannot be a value of constancy, because this would imply that for some $\delta > 0$, $P(m + \delta) = 0$, which contradicts the fact that m is the essential lower bound of $f(\theta)$.

A single-valued function has an inverse if it is univalent. Delete from $[m, M]$ each interval of constancy, with the exceptions of the left hand end point of each interval, and let D designate the resulting set. Let $P(y)$ be the restriction of $P(y)$ to D. It will be univalent on D and its range is still the set R. Let $\bar{f}^*(\theta)$ be the inverse of $\bar{P}(y)$. Its domain of definition is R and its range of values is D.

Let θ_1, θ_2 be any two points in R. Then there are unique, distinct values of y in D, say y_1, y_2, such that $P(y_1) = \theta_1$, $P(y_2) = \theta_2$ and $\bar{f}^*(\theta_1) = y_1$, $\bar{f}^*(\theta_2) = y_2$. Since, from (4.2),

$$\frac{P(y_2) - P(y_1)}{y_2 - y_1} > 0,$$

it follows that

$$\frac{\theta_2 - \theta_1}{\bar{f}^*(\theta_2) - \bar{f}^*(\theta_1)} > 0,$$

so that $\bar{f}^*(\theta)$ is an increasing function on R.

We next define a function $f^*(\theta)$ which extends the definition of $\bar{f}^*(\theta)$ to the interval $[0, \pi]$. For $\theta \in R$, let $f^*(\theta) = \bar{f}^*(\theta)$. If $\theta \neq R$ but does lie in $[0, \pi]$, it either lies in $[0, \varepsilon)$ or in an interval A_i. In the first case we let $f^*(\theta) = f^*(\varepsilon) = m$; in the second case we let $f^*(\theta) = \bar{f}^*(a_i^!)$. The resulting function $f^*(\theta)$ is a non-decreasing function on $[0, \pi]$. We now note that

$$f^*(0) = m \qquad (f^*(\pi) = M). \tag{4.4}$$

The second follows from (4.3). The first follows from the above construction, with the added observation already made that zero is not a value of constancy.

Since $f^*(\theta)$ is non-decreasing on $[0, \pi]$,

$$\text{meas } E(f^*(\theta) \leq y) = \theta(y) \qquad (m \leq y \leq M),$$

where $\theta(y)$ is the least upper bound of the values of θ which satisfy $f^*(\theta) \leq y$. We now plan to show that $\theta(y) = P(y)$, thus showing that $f^*(\theta)$ is equimeasurable with $f(\theta)$.

Let y be a value assumed by $f^*(\theta)$. It is also assumed by $\bar{f}^*(\theta)$, and $P(y)$ is the unique solution of $\bar{f}^*(\theta) = y$. If $f^*(\theta) = y$ has other solutions, they are all less than $P(y)$, so $\theta(y) = P(y)$.

Suppose y is not assumed by $f^*(\theta)$. It then belongs to one of the intervals B_i. Thus

$$P(y) = P(b_i) \tag{4.5}$$

and

$$E(f^*(\theta) \leq y) = E(f^*(\theta) \leq b_i). \tag{4.6}$$

The value b_i is on the range of $f^*(\theta)$ and so by the considerations of the previous paragraph

$$\text{meas } (f^*(\theta) \leq b_i) = P(b_i). \tag{4.7}$$

Combining (4.5), (4.6) and (4.7) then yields

$$\text{meas } E(f^*(\theta) \leq y) = P(y).$$

This completes the proof.

Proof of Lemma 2. We will actually show that

$$\int_0^{\pi} F(f(\theta))\, d\theta = \int_m^M F(t)\, dP(t), \tag{4.8}$$

where $P(t)$ is the measure function defined in (4.1). Since $f^*(\theta)$ has the same measure function as $f(\theta)$, the result (2.7) then follows.

First assume that $0 \leq m < M < \infty$. The integral

$$\int_0^{\pi} f^k(\theta)\, d\theta\,, \tag{4.9}$$

with k a positive integer, is defined as a limit of sums

$$\sum_{i=1}^{n} y_{i,n} \text{ meas } E(y_{i-1,n} < f^k(\theta) \leq y_{i,n})\,, \tag{4.10}$$

where

$$m^k = y_{0,n} < y_{1,n} < \cdots < y_{n,n} = M^k$$

and

$$\lim_{n \to \infty} \max_{i=1,\cdots,n} \{|y_{i,n} - y_{i-1,n}|\} = 0\,.$$

Display (4.10) can also be written

$$\sum_{i=1}^{n} t_{i,n}^k (P(t_i) - P(t_{i-1}))\,, \tag{4.11}$$

where

$$t_{i,n} = y_{i,n}^{1/k} \qquad (i = 1, \cdots, n)\,.$$

In the limit (4.11) has the value

$$\int_m^M t^k\, dP(t)\,. \tag{4.12}$$

Given any $\varepsilon > 0$, and any continuous function $F(x)$ defined on $[m, M]$, there are polynomials $P_1(x)$, $P_2(x)$ which satisfy

$$F(x) - \frac{\varepsilon}{\pi} < P_1(x) < F(x) < P_2(x) < F(x) + \frac{\varepsilon}{\pi} \tag{4.13}$$

on the interval $[m, M]$. We then obtain a chain of inequalities, using the fact that (4.9) equals (4.12) and using (4.13); namely,

$$\int_0^{\pi} F(f(\theta))\, d\theta - \varepsilon < \int_0^{\pi} P_1(f(\theta))\, d\theta = \int_m^M P_1(t)\, dP(t)$$

$$< \int_m^M F(t)\, dP(t) < \int_m^M P_2(t)\, dP(t) = \int_m^M P_2(f(\theta))\, d\theta$$

$$< \int_0^{\pi} F(f(\theta))\, d\theta + \varepsilon\,.$$

Since this is true for all $\varepsilon > 0$, we find (4.8) to be true if $m \geq 0$.

Next, suppose $m < 0$. The function $f(\theta) + |m|$ is non-negative, has measure function $P_1(y) = P(y - |m|)$, and has range $[0, M + |m|]$. If $F(x)$ is a continuous function on $[m, M]$, then

$$G(x) = F(x + m) \tag{4.14}$$

is continuous on $[0, M + |m|]$. Hence

$$\int_0^{\pi} G(f(\theta) + |m|)\, d\theta = \int_0^{M+m} G(t)\, dP_1(t)\,.$$

Using (4.14) and the definition of $P_1(y)$, we find

$$\int_0^{\pi} F(f(\theta))\, d\theta = \int_0^{M+m} F(t + m)\, dP(t - |m|) = \int_m^{M} F(t)\, dP(t)\,.$$

This completes the proof.

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