PERIODIC, ALMOST-PERIODIC, AND SEMIPERIODIC SEQUENCES

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Dedicated by the second author to his teacher, C. W. Walmsley, who died March 27, 1962, in Derbyshire, England.

1. INTRODUCTION

We acknowledge valuable consultations with G. Rayna and G. A. Stengle.

This article was suggested by two distinct lines of investigation, which merged after each had proceeded independently. These were the study of almost-periodic functions and sequences, and the study of matrix transformations of periodic sequences initiated by Vermes [6], [7] and Newton [5]. In the latter investigation, the the natural desire of functional analysts to work in a complete space (see, for example, Corollary 2) prompted us to deal with the completion of the space of periodic sequences. This completion turns out to be smaller than the space of almost-periodic sequences, thus damping any hopes of an extensive use of the theories of Bohr and von Neumann. The situation is this; each almost-periodic function on $(-\infty, \infty)$ is uniformly approximable by linear combinations of periodic functions. In attempting to restrict this result to the integers, we run into several snags. First, the restriction of a periodic function need not be a periodic sequence; second, the set of periodic sequences is already a linear space, so that its linear closure is the same as its closure; and, finally, that closure is actually smaller than the space of restrictions of almost-periodic functions, and therefore the analogous theorem is actually false.

A sequence x of complex numbers is called *semiperiodic* if to each $\varepsilon > 0$, there corresponds a positive integer n such that $|x_k - x_{k+rn}| < \varepsilon$ for all r, k; it is called *almost-periodic* if to each $\varepsilon > 0$, there corresponds a positive integer n such that every interval $(K, K + n)$, $K = 1, 2, \cdots$, contains an integer t satisfying the condition $|x_r - x_{r+t}| < \varepsilon$ for all r.

Taking m to be the familiar Banach space of all bounded complex sequences with $\|x\| = \sup |x_n|$, the closure in m of the set p of periodic sequences is shown, in Theorem 1, to be precisely the set q of semiperiodic sequences.

A related problem of interest concerns the properties of the space q as a subspace of m, considered as the conjugate of the space ℓ of absolutely convergent series with $\|x\| = \Sigma |x_n|$. The space q turns out to be large enough to be norming over ℓ. (See Section 5, below, for a definition of this concept.)

Since q is so large, it is interesting that there exist regular matrices which sum all of its sequences (for example, the Cesàro matrix, which yields the von Neumann mean) especially since known theorems place restrictions on the number of bounded divergent sequences summed by a regular matrix.

If S denotes the set of (C, 1)-summable bounded sequences, then $S \setminus c_0$ is norming over ℓ: this suggests the problem of identifying the matrices A having the

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property that $(c_A \cap m) \setminus c_0$ is norming over ℓ. It seems plausible that every conull matrix has this property.

2. PERIODIC SEQUENCES

In [7], Vermes asks whether p has a complete norm. We can immediately answer this in the negative, since p has a countable Hamel basis and is therefore of first category in itself (it is the union of an increasing sequence of finite-dimensional subspaces, hence the union of an increasing sequence of closed subspaces). The existence of the countable basis is observed in [6]; we shall exhibit one explicitly, since we can fortunately show that it is a Schauder basis for q.

3. SEMIPERIODIC SEQUENCES

The sequence $\{\cos n\}$ is not uniformly approximable by periodic sequences: if x is a sequence of period k, then, for suitable n,

$$|x_{nk} - \cos nk| = |x_k - \cos nk| > 1/2.$$

This shows that $\bar{p}$, the closure of p in m, is not the set of almost-periodic sequences.

THEOREM 1. $\bar{p} = q$, *that is, the uniform closure of the set of periodic sequences is the set of semiperiodic sequences.*

Let x be semiperiodic. Let $\varepsilon > 0$ be given, and choose n as in the definition of semiperiodicity. If y is defined by $y_k = x_k$ for $k \leq n$, and $y_{k+n} = y_k$, then $y \in p$ and clearly $\|x - y\| < \varepsilon$.

Conversely, let $x \in \bar{p}$ and $\varepsilon > 0$. There exists $y \in p$ with $\|x - y\| < \varepsilon/2$. Let n be the period of y. Then, for any r and k,

$$\begin{aligned}|x_k - x_{k+rn}| &\leq |x_k - y_k| + |y_k - x_{k+rn}| = |x_k - y_k| + |y_{k+rn} - x_{k+rn}| \\ &\leq 2\|x - y\| < \varepsilon.\end{aligned}$$

Thus x is semiperiodic.

4. A SCHAUDER BASIS

A *Schauder basis* for a space is a sequence such that each member of the space is a unique infinite linear combination of its terms.

Corresponding to any pair a, b of positive integers with $b \leq a$, we denote by $d_{a,b}$ that sequence which has period a and whose first a elements are zero, except for an element 1 in the b^{th} place. For example, $d_{1,1} = \{1\}$ and

$$d_{4,3} = \{0, 0, 1, 0, 0, 0, 1, 0, \cdots\}.$$

Let

$$H = \{d_{1,1}, d_{2,2}, d_{6,3}, d_{6,4}, d_{6,5}, d_{6,6}, d_{24,7}, \cdots\},$$

where the terms are $d_{n!,k}$ for $(n-1)! < k \leq n!$. In the matrix whose rows are the members of H, the main diagonal consists of ones, and all terms below the main diagonal are zero. From this we see immediately that H is a Hamel basis for p.

We write $H = \{h_n\}$; for example, $h_1 = d_{1,1}$, $h_4 = d_{6,4}$.

It is important to observe that if

$$x = \{x_1, x_2, \cdots, x_n, x_1, x_2, \cdots\} \in p \quad \text{and} \quad x = \sum_{k=1}^{n} a_k h_k,$$

then each a_r $(1 \leq r \leq n)$ is determined by $x_1, x_2, \cdots, x_r$ and is independent of $x_{r+1}, \cdots, x_n$. We shall use this fact.

H has a strong separation property; namely, for each $x \in H$, the distance from x to the linear span of $H \setminus \{x\}$ is 1, since no two members of H have the same number of 0's before their first 1. This implies that each element of q has at most one expression as an infinite linear combination of elements in H. Indeed, the Hahn-Banach Theorem supplies a biorthogonal set of functionals.

Thus, finally, H will be shown to be a Schauder basis for q when we have proved that every $x \in q$ is an infinite linear combination of elements in H.

Let $x \in q$ be given, and let the sequence $\{a_k\}$ be defined by the requirement that for each n the j^{th} term of $\Sigma_{k=1}^{n} a_k h_k$ is x_j for $j = 1, 2, \cdots, n$. Since, as remarked above, each a_k depends only on $x_1, x_2, \cdots, x_k$, this requirement well-defines $\{a_k\}$. Observe that the period of $\Sigma_{k=1}^{n} a_k h_k$ is not in general equal to n. Indeed, if $(p-1)! < n \leq p!$, then $p!$ is a period, and possibly the smallest period.

We now show that $\lim_{n\to\infty} \Sigma_{k=1}^{n} a_k h_k = x$. Let $\varepsilon > 0$ be given. By definition, there exists a sequence s of period r (hence of period rk for each positive integer k) such that $\|s - x\| < \varepsilon$.

First consider the case $n = p!$, where r divides $p!$. Here

$$\sum_{k=1}^{n} a_k h_k = \{x_1, x_2, \cdots, x_n, x_1, x_2, \cdots, x_n, \cdots\},$$

since x has period $p!$ (recall that for $k \leq p!$, h_k has period $p!$) and the first $p!$ terms are completely determined by $\{a_k\}_{k=1}^{p!}$.

Since s has period $p!$ and $\|s - x\| < \varepsilon$, we see that

$$\|s - \sum_{k=1}^{n} a_k h_k\| < \varepsilon.$$

Hence $\|\Sigma_{k=1}^{n} a_k h_k - x\| < 2\varepsilon$.

Next we consider the case $n = p! + q$, where $0 < q < (p+1)! - p!$. We have already established that

$$\|x - \sum_{k=1}^{p!} a_k h_k\| < 2\varepsilon \quad \text{and} \quad \|x - \sum_{k=1}^{(p+1)!} a_k h_k\| < 2\varepsilon.$$

Observe that for $i \neq j$ and $p! < i, j < (p+1)!$, the relation $(h_i)_k = 1$ implies that $(h_j)_k = 0$ for each k; in other words, for such i, j, the two rows h_i and h_j never have 1's in the same column. Hence, for $p! < k < (p+1)!$, we have $|a_k| < 2\varepsilon$, and therefore

$$\left\| \sum_{k=p!+1}^{n} a_k h_k \right\| < 4\varepsilon \qquad \text{for } p! < n < (p+1)!.$$

Thus

$$\left\| x - \sum_{k=1}^{n} a_k h_k \right\| < 6\varepsilon$$

for $p! < n < (p+1)!$ and $\| x - \Sigma_{k=1}^{(p+1)!} a_k h_k \| < 2\varepsilon$.

But $p!$ is any factorial such that r (the period of s) divides $p!$. Hence $\| x - \Sigma_{k=1}^{n} a_k h_k \| < 6\varepsilon$ for all $n \geq p!$. We have thus proved the following result.

THEOREM 2. *H is a Schauder basis for* q.

5. NORMING SUBSPACES

Let B be a normed space, and S a total subset of B^*. Then S is a set of continuous linear functions on B such that $x = 0$ whenever $f(x) = 0$ for all $f \in S$. A new norm for B is defined by

$$\|x\|_S = \sup \{ |f(x)| / \|f\| : f \in S, f \neq 0 \}.$$

If this new norm is equivalent to the original norm for B, S is called *norming*. An example of a non-norming (total) subspace of $\ell = c_0^*$ is given in [3] (see the 13 lines beginning with the last line on p. 1067).

THEOREM 3. q *is a norming subspace of* $\ell^* = m$. *Indeed,* $\|y\|_q = \|y\|$ *for* $y \in \ell$.

Let $y \in \ell$ and $\|y\| = \Sigma |y_k| = 1$. Let n be a positive integer, and x a sequence of period n such that $x_k = \operatorname{sgn} y_k$ for $1 \leq k \leq n$. Then $x \in p$, hence $x \in m$; also, $\|x\| = \sup |x_k| \leq 1$, and we may consider $x \in \ell^*$ by writing $x(y) = \Sigma x_k y_k$. Then $x(y) = \Sigma_{k=1}^{n} |y_k| + R$, where

$$|R| = \left| \sum_{k=n+1}^{\infty} x_k y_k \right| \leq \sum_{k=n+1}^{\infty} |y_k| \to 0$$

as $n \to \infty$. Thus $\|y\|_q \geq 1$ and therefore $\|y\|_q = 1 = \|y\|$.

COROLLARY 1. *The set* S *of bounded sequences summed by the* $(C, 1)$ *method has the property that* $S \setminus C_0$ *is norming over* ℓ.

We ask whether every regular matrix that sums a bounded divergent sequence has this property. A result in this direction is that of Agnew [1], which asserts that such a matrix must sum a non-separable subset of m. We also conjecture that every co-null matrix has this property.

COROLLARY 2. *If a sequence* a *has the property that* $\Sigma a_n x_n$ *converges for all* $x \in q$, *then* $a \in \ell$. *This result is false if* q *is replaced by* p.

For each positive integer n, define a functional f_n on q by $f_n(x) = \Sigma_{k=1}^n a_k x_k$. We may identify f_n with $\{a_1, a_2, \cdots, a_n, 0, 0, \cdots\} \in \ell$. By Theorem 3, it follows that $\|f_n\| = \Sigma_{k=1}^n |a_k|$, the latter being the norm of f_n as a function on m. Since $\lim_{n\to\infty} f_n(x)$ exists for each x, and since q is complete, it follows from the uniform boundedness principle that $\|f_n\|$ is bounded.

That the result is false for p follows from the existence (see [7]) of annihilators of p. If $\Sigma a_n x_n$ were 0 for all $x \in p$, and if a were in ℓ, it would follow that $\Sigma a_n x_n$ vanishes on all of q, since it is continuous. By Theorem 3, this implies that $a = 0$.

We make the usual remark that in Corollary 2 it suffices to assume that the series is bounded rather than convergent.

COROLLARY 3. q *has no annihilators.*

As just mentioned, p has annihilators.

6. THE CONJUGATE SPACE OF q

The space q^* contains functions not expressible as $\Sigma a_n x_n$; for example, $\lim x_{n!}$ is defined on q but is not of that form. Thus $q^* \neq \ell$.

Since q is a separable subspace of m, it is known that each member of q^* is expressible in terms of a matrix. See [2, pp. 68-72] and [4] for an alternative proof.

We have an explicit construction that gives a canonical matrix for a given member of q^*. In elegance, it leaves something to be desired, and we shall not present it here.

7. MATRICES SUMMING q

A corollary of Theorem 3 is the standard boundedness condition for matrices, namely,

THEOREM 4. *If a matrix* A *sums all the members of* q, *it must satisfy* $\|A\| < \infty$, *where* $\|A\| = \sup_n \Sigma_k |a_{nk}|$.

Since q is complete, we can apply standard techniques of functional analysis. In the first place, Ax exists for each $x \in q$. Thus, for each n, $\Sigma a_{nk} x_k$ is convergent, hence by Corollary 2, $\Sigma_k |a_{nk}| < \infty$ for each n.

For $n = 1, 2, \cdots$, define a functional g_n on q by $g_n(x) = \Sigma_k a_{nk} x_k$. By Theorem 3, $\|g_n\| = \Sigma_k |a_{nk}|$. Finally, since $\lim g_n(x)$ exists for all $x \in q$, the result follows by the uniform boundedness principle.

Since q is disjoint from c_0, it is not surprising that matrices summing q need not be conservative. An example is the matrix (a_{nk}) with

$$a_{n1} = (-1)^n, \quad a_{n,n!+1} = (-1)^{n+1}, \quad a_{nk} = 0 \text{ otherwise.}$$

This matrix sums all members of q to 0; that is, $\lim_{n\to\infty} \Sigma_{k=1}^\infty a_{nk} x_k = 0$ for all $x \in q$.

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