THE SINGULARITIES OF CONTINUOUS FUNCTIONS AND VECTOR FIELDS

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1. INTRODUCTION

The singular points of a system of ordinary differential equations

$$\frac{dx_i}{dt} = f_i(x_1, \cdots, x_n) \qquad (i = 1, \cdots, n) \tag{1}$$

are usually defined as the points at which all f_i are 0. Much attention has been devoted to the study of the solutions in the neighborhood of such points, a powerful tool being the approximation of the f_i by polynomials on the basis of Taylor's series or formula. However, it is also natural to use the term *singular point* for a point at which the existence theorem is inapplicable for some reason, for example, because the f_i fail to remain continuous at the point. The approximation by polynomials is then not available, and one is apparently faced with an entirely different problem.

It is the purpose of the present paper to show that the second problem can be reduced to the first. Our procedure is based on a geometric interpretation of the differential equations: One is given a vector field; the solutions sought are to have the given vectors as tangents. The length of the vectors is immaterial. Hence we seek to modify the length by multiplication by a positive scalar in such a fashion as to force the length to approach 0 at the singular point. The new vector field has then a singular point of the conventional type. We shall in fact show that if the f_i are of class $C^{(m)}$ in a deleted neighborhood of the singular point, then the modified vector field can be chosen to be of class $C^{(m)}$ in a full neighborhood of the singular point. Hence analytical tools such as Taylor's formula become available.

The modification of the vector field is equivalent to the introduction of a new parameter, replacing t, to regularize the problem. The process thus resembles Sundman's famous regularization of collisions in the problem of three bodies.

2. A LEMMA ON $C^{(\infty)}$ EXTENSIONS OF A NULL FUNCTION

We first establish a lemma which provides a smooth function resembling the distance from a given closed set in n-space.

LEMMA 1. *Let* E *be a closed proper subset of Euclidean* n-*space*, R_n. *There exists a function* $f(x_1, \cdots, x_n)$ *of class* $C^{(\infty)}$ *in* R_n, *such that* f *and all its partial derivatives are identically* 0 *on* E *and* f *is greater than* 0 *on* $D = R_n - E$.

Proof. We represent D as a union of open n-dimensional balls D_h $(h = 1, 2, \cdots)$. Let ρ_h be the radius of D_h, and let $r_h(x)$ be the distance of $x = (x_1, \cdots, x_n)$ from the center of D_h. Let $f_h(x)$ be defined in R_n as follows:

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$$f_h(x) = \exp[-\{r_h^2(x) - \rho_h^2\}^{-2}] \qquad (x \in D_h),$$

$$f_h(x) = 0 \qquad (x \notin D_h).$$

Then $f_h(x)$ is of class $C^{(\infty)}$ in R_n and $f_h(x) > 0$ in D_h. We can further choose constants k_{hs} ($s = 0, 1, 2, \cdots$) such that each partial derivative of f_h of order s is in absolute value less than k_{hs} for all x in R_n. We set

$$a_h = [2^h \max(k_{h0}, k_{h1}, \cdots, k_{h,h-1})]^{-1},$$

so that $0 < a_h k_{hs} \leq 2^{-h}$ for $s < h$. For each s, the series $\Sigma_1^\infty a_h k_{hs}$ is then convergent, since each term after the s-th is less than the corresponding term in the series $\Sigma_1^\infty 2^{-h}$. The series $\Sigma a_h f_h$ is dominated by $\Sigma a_h k_{h0}$, and hence it converges uniformly to a function $f(x)$. For fixed $j_1, \cdots, j_n$ and $s = j_1 + \cdots + j_n$, the series of partial derivatives

$$\sum_{h=1}^{\infty} a_h \frac{\partial^s f_h}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}$$

converges uniformly in R_n. Hence f is of class $C^{(\infty)}$. By the construction of the f_h, $f > 0$ in D, $f = 0$ in E, and all partial derivatives of f are 0 in E.

Remark. The lemma is related to the extension theorems of Whitney (see [3], for example), but it does not seem to be included in them.

3. QUOTIENT REPRESENTATION OF CONTINUOUS FUNCTIONS WITH SINGULARITIES

We show now that every function $F(x)$, continuous in R_n except on a compact set E, can be represented in the form $F = F_1/F_2$, where F_1 and F_2 are continuous throughout R_n and $F_1 = F_2 = 0$ on E. This is suggestive of the representation of a meromorphic function as a quotient of two entire functions. We show also that "continuous" can be replaced by "of class $C^{(m)}$ for $m = 1, 2, \cdots, \infty$."

THEOREM 1. *Let* $F(x)$ *be continuous in* R_n *except on a compact set* E. *Then there exist functions* F_1 *and* F_2, *continuous throughout* R_n, *such that*

$$F = F_1/F_2 \text{ on } D = R_n - E, \qquad F_1 = F_2 = 0 \text{ on } E, \qquad F_2 > 0 \text{ on } D.$$

Proof. We first choose a ball G: $\Sigma_1^n x_i^2 \leq R^2$ such that E is contained in the interior of G. Let f be chosen as in Lemma 1 relative to E. Let

$$r_0 = \text{l.u.b.}\,\{f(x) \mid x \in G\},$$

so that $r_0 > 0$. Let K_r be the set $\{x \mid f(x) \geq r,\ x \in G\}$, and let

$$\psi_0(r) = \begin{cases} 1 + \text{l.u.b.}\,\{|F(x)| \mid x \in K_r\} & (0 < r \leq r_0), \\ \psi_0(r_0) & (r > r_0). \end{cases}$$

Then $\psi_0(r)$ is a nonincreasing function of r and $\psi_0(r) \geq 1$ for $0 < r < \infty$. Set

$$\mu(r) = \int_0^r \frac{1}{\psi_0(u)}\,du \quad \text{for } r > 0, \qquad \mu(0) = 0.$$

Then $\mu(r)$ is a continuous function of r for $0 \leq r < \infty$, and $\mu(r) > 0$ for $r > 0$. Set

$$F_2(x) = \mu[f(x)],$$

$$F_1(x) = \begin{cases} 0 & (x \text{ in } E), \\ F(x) \cdot F_2(x) & (x \text{ in } D = R_n - E). \end{cases}$$

Then, on E, $f(x) = 0$, so that $F_2(x) = 0$; $F_1(x) = 0$ in E by definition. Since μ is continuous, F_2 is continuous in R_n, and $F_2(x) > 0$ in D. Now

$$\mu(r) \leq r/\psi_0(r) \qquad (r > 0).$$

Hence on the locus $f = r$ in G $(r > 0)$,

$$|F_1(x)| \leq |F(x)| \frac{r}{\psi_0(r)} \leq r.$$

Accordingly, $|F_1(x)| \leq f(x)$ in G. It follows immediately that $F_1(x)$ is continuous at each point of E, and hence $F_1(x)$ is continuous in R_n. Since $F_2(x) > 0$ in D, $F(x) = F_1(x)/F_2(x)$ in D, as asserted.

THEOREM 2. *If in Theorem* 1 $F(x)$ *is of class* $C^{(m)}$ *in* D (m *a positive integer*), *then* $F_1(x)$, $F_2(x)$ *can be chosen to be of class* $C^{(m)}$ *in* R_n.

Proof. We proceed as in the proof of Theorem 1 and let

$$\psi_m(r) = 1 + \max_{x \in K_r} \left\{ |F| + \sum \left| \frac{\partial^{i_1+\cdots+i_n} F}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} \right| \right\} \qquad (0 < r \leq r_0), \tag{2}$$

where the summation is over all $i_1, \cdots, i_n$ such that $1 \leq i_1 + \cdots + i_n \leq m$. For $r > r_0$, we set $\psi_m(r) = \psi_m(r_0)$. Let $\mu_m(r)$ be defined inductively:

$$\mu_0(r) = \int_0^r \frac{1}{\psi_m(u)}\,du, \qquad \mu_k(r) = \int_0^r \mu_{k-1}(u)\,du \quad (k = 1, \cdots, m).$$

Then $\mu_k(r)$ is of class $C^{(k)}$ for $r \geq 0$, and

$$0 < \mu_k(r) \leq \frac{r^{k+1}}{\psi_m(r)} \qquad (k = 0, 1, \cdots, m,\ r > 0). \tag{3}$$

We let $F_2(x) = \mu_m[f(x)]$, again let $F_1(x) = 0$ for x in E, and let

$$F_1(x) = F(x) \cdot F_2(x)$$

for x in D. It then follows that $F_2(x)$ is of class $C^{(m)}$ in R_n and $F_2(x) > 0$ in D; accordingly, $F_1(x)$ is of class $C^{(m)}$ in D. On the locus $f = r$ in K_r $(r > 0)$,

$$|F_1(x)| \leq |F(x)| \cdot \mu_m(r) \leq |F(x)| \frac{r^{m+1}}{\psi_m(r)} \leq r^{m+1}.$$

Accordingly, $|F_1(x)| \leq [f(x)]^{m+1}$ in G. It follows that $F_1(x)$ is continuous at each point of E, and hence $F_1(x)$ is continuous in R_n.

Furthermore, grad $F_1 = 0$ on E. For if x is in E and $\|\Delta x\|$ is the Euclidean norm of the vector $\Delta x = (\Delta x_1, \cdots, \Delta x_n)$, then for positive and sufficiently small $\|\Delta x\|$,

$$\left|\frac{F_1(x + \Delta x) - F_1(x)}{\|\Delta x\|}\right| = \left|\frac{F_1(x + \Delta x)}{\|\Delta x\|}\right|$$

$$\leq \frac{f(x + \Delta x)}{\|\Delta x\|} [f(x + \Delta x)]^m$$

$$\leq \left|\frac{f(x + \Delta x) - f(x)}{\|\Delta x\|}\right| [f(x + \Delta x)]^m.$$

As $\|\Delta x\| \to 0$, both of the last two factors approach 0, since $f = 0$ on E and grad $f = 0$ on E. Therefore grad $F_1 = 0$ on E.

Next, on the locus $f = r > 0$ in G,

$$\|\text{grad } F_1\| = \|F \text{ grad } F_2 + F_2 \text{ grad } F\|$$

$$\leq |F| \, \|\text{grad } \mu_m(f)\| + |\mu_m(f)| \, \|\text{grad } F\|$$

$$\leq |F| \, \mu'_m(r) \, \|\text{grad } f\| + \frac{r^{m+1}}{\psi_m(r)} \|\text{grad } F\|$$

$$\leq |F| \, \mu_{m-1}(r) \, \|\text{grad } f\| + \frac{r^{m+1}}{\psi_m(r)} \|\text{grad } F\|$$

$$\leq |F| \frac{r^m}{\psi_m(r)} \|\text{grad } f\| + \frac{r^{m+1}}{\psi_m(r)} \|\text{grad } F\|.$$

Therefore, everywhere in G,

$$\|\text{grad } F_1\| \leq f^m \|\text{grad } f\| + f^{m+1}. \tag{4}$$

Since $f = 0$ and grad $f = 0$ in E, we conclude that grad $F_1 \to 0$ as x approaches a point of E. Therefore F_1 is of class C' on E and hence in R_n.

For $m \geq 2$, we consider the second derivatives of F_1. By (4), each first derivative F_{1x_j} satisfies an inequality

$$|F_{1x_j}| \leq f^m \|\text{grad } f\| + f^{m+1} \qquad (m \geq 2, \ x \in G)$$

and is 0 on E. Hence, as above, we find that each difference quotient of F_{1x_j} at a point of E is in absolute value less than or equal to the absolute value of the corresponding difference quotient of f times $f^{m-1} \|\text{grad } f\| + f^m$ evaluated at $x + \Delta x$.

Again both factors approach 0 as $\|\Delta x\| \to 0$, and we conclude that all second derivatives of F_1 are 0 on E.

Next, on the locus $f = r > 0$ in K_r,

$$|F_{1x_jx_k}| = |F_{x_j}F_{2x_k} + F_{x_k}F_{2x_j} + F_{x_jx_k}F_2 + F F_{2x_jx_k}|$$

$$\leq |F_{x_j}|\,\mu_{m-1}(r)\,|f_{x_k}| + |F_{x_k}|\,\mu_{m-1}(r)\,|f_{x_j}| + |F_{x_jx_k}|\,\mu_m(r)$$

$$+ |F|\,\{\mu_{m-2}(r)\,|f_{x_j}|\cdot|f_{x_k}| + \mu_{m-1}(r)\,|f_{x_jx_k}|\}$$

$$\leq \frac{1}{\psi_m(r)}(|F_{x_j}|\,r^m\,|f_{x_k}| + |F_{x_k}|\,r^m\,|f_{x_j}| + |F_{x_jx_k}|\,r^{m+1}$$

$$+ |F|\,\{r^{m-1}\,|f_{x_j}|\cdot|f_{x_k}| + r^m\,|f_{x_jx_k}|\})$$

by (3). Since, by (2), $\psi_m(r)$ exceeds the absolute value of F and all its first and second derivatives on the locus $f = r$, we conclude that

$$|F_{1x_jx_k}| \leq f^{m-1}|f_{x_j}|\cdot|f_{x_k}| + f^m(|f_{x_k}| + |f_{x_j}| + |f_{x_jx_k}|) + f^{m+1}.$$

Hence each second partial derivative approaches 0 as x approaches a point of E. Thus the second partial derivatives are continuous.

The third, fourth, ···, m-th partial derivatives are analyzed in similar fashion, and we conclude that F_1 is of class $C^{(m)}$ everywhere.

THEOREM 3. *If in Theorem* 1 F(x) *is of class* $C^{(\infty)}$ *in* D, *then* F_1, F_2 *can be chosen to be of class* $C^{(\infty)}$ *in* R_n.

Proof. We require a lemma:

LEMMA 2. *Let* $\{\mu_m(r)\}$ (m = 0, 1, 2, ···) *be a sequence of functions of* r *on the interval* $0 \leq r < \infty$ *such that* $\mu_m(r)$ *is continuous,* $\mu_m(0) = 0$, *and* $\mu_m(r) > 0$ *for* $r > 0$. *Then there exists a sequence of numbers* $r_m > 0$ *and a function* $\mu(r)$ *of class* $C^{(\infty)}$ *such that*

$$\mu^{(m)}(0) = 0 \ (m \geq 0), \qquad \mu(r) > 0 \ (r > 0), \qquad |\mu^{(m)}(r)| < \mu_m(r) \ (0 < r \leq r_m).$$

Proof. For k = 1, 2, ··· choose $\delta_k(r)$, a function of class $C^{(\infty)}$ for $0 \leq r < \infty$, such that $\delta_k(r) > 0$ for $2^{1-2k} < r < 2^{3-2k}$ and $\delta_k(r) = 0$ otherwise; choose $\varepsilon_k(r)$, a function of class $C^{(\infty)}$ for $0 \leq r < \infty$, such that $\varepsilon_k(r) > 0$ for $2^{-2k} < r < 2^{2-2k}$ and $\varepsilon_k(r) = 0$ otherwise. Let

$$\Delta(r) = \sum_1^\infty a_k\,\delta_k(r), \qquad E(r) = \sum_1^\infty b_k\,\varepsilon_k(r),$$

where $\{a_k\}$ and $\{b_k\}$ are sequences of positive numbers to be specified. Let $\eta(r)$ be a function of class $C^{(\infty)}$ for $0 \leq r < \infty$, equal to 0 for $\leq r \leq 1$ and positive otherwise. The function $\mu(r)$ is then defined as the sum

$$\mu(r) = \eta(r) + \Delta(r) + E(r).$$

For each $r > 0$ at least one of the three functions on the right is positive, so that $\mu(r) > 0$. The three functions are of class $C^{(\infty)}$ for $r > 0$; if the a_k and b_k are sufficiently small, they are of class $C^{(\infty)}$ for $r \geq 0$, with all derivatives 0 at $r = 0$. This is proved in the same way as Lemma 1.

Next, for proper choice of the sequence $\{a_k\}$,

$$\delta(r) < \frac{1}{2}\mu_0(r) \quad (r \in [2^{-1}, 2]), \quad \delta(r) < \frac{1}{2}\mu_0(r) \text{ and } |\delta'(r)| < \frac{1}{2}\mu_1(r) \quad (r \in [2^{-3}, 2^{-1}]),$$

and

$$|\delta^{(j)}(r)| < \frac{1}{2}\mu_j(r) \qquad (j = 0, 1, \cdots, k;\ r \in [2^{1-2k}, 2^{3-2k}]),$$

so that

$$|\delta^{(k)}(r)| < \frac{1}{2}\mu_k(r) \qquad (0 < r \leq 2^{3-2k}).$$

Similarly, for proper choice of the b_k,

$$|\varepsilon^{(k)}(r)| < \frac{1}{2}\mu_k(r) \qquad (0 < r \leq 2^{2-2k}).$$

Since $\eta^{(k)}(r) = 0$ for $0 \leq r \leq 1$, we conclude that

$$|\mu^{(k)}(r)| < \mu_k(r) \qquad (0 < r \leq 2^{-2k};\ k = 0, 1, 2, \cdots).$$

Thus the lemma is established.

Now, to prove Theorem 3, we define $\psi_m(r)$ $(m = 0, 1, 2, \cdots)$ as in the proofs of Theorems 1 and 2, and we let

$$\mu_m(r) = \mu_{m0}(r) = \int_0^r \frac{1}{\psi_m(u)}\,du, \qquad \mu_{m1}(r) = \int_0^r \mu_{m0}(u)\,du, \cdots,$$

so that, as in the proof of Theorem 2,

$$0 < \mu_{mk}(r) < \frac{r^{k+1}}{\psi_m(r)} \qquad (k = 0, 1, \cdots, m;\ r > 0). \tag{5}$$

We then choose $\mu(r)$ in accordance with Lemma 2. Hence, for some r_0, $\mu(r) < \mu_0(r) = \mu_{00}(r)$ if $0 < r < r_0$. Next, for $0 < r < r_1$,

$$|\mu(r)| = \left|\int_0^r \mu'(u)\,du\right| < \int_0^r \mu_1(u)\,du = \int_0^r \mu_{10}(u)\,du = \mu_{11}(r),$$

and in general, for $0 < r < r_m$,

$$|\mu^{(m)}(r)| < \mu_m(r) = \mu_{m0}(r),$$

(6) $$|\mu^{(m-1)}(r)| = \left|\int_0^r \mu^{(m)}(u)\,du\right| < \int_0^r \mu_{m0}(u)\,du = \mu_{m1}(r), \cdots,$$

$$|\mu^{(m-k)}(r)| < \mu_{mk}(r) \qquad (k = 0, 1, \cdots, m).$$

We again let $F(x) = \mu[f(x)]$, $F_1(x) = 0$ in E, $F_1(x) = F(x)\,F_2(x)$ in D. Then $F_2(x)$ is of class $C^{(\infty)}$ in R_n, and $F_2(x) = 0$ if and only if $x \in E$. Because of (5) and (6), the proof of Theorem 2 can now be repeated without essential change to show that, for each m, $F_1(x)$ is of class $C^{(m)}$; that is, $F_1(x)$ is of class $C^{(\infty)}$.

4. GENERALIZATIONS

Instead of R_n we can consider an n-dimensional differentiable manifold M_n. If M_n is of class $C^{(\infty)}$, Lemma 1 can be generalized to M_n with only minor modifications in the proof. If M_n is only of class $C^{(m)}$ (m finite), one can prove a lemma analogous to Lemma 1 for M_n, but with f(x) of class $C^{(m)}$ only. Theorem 1 can then be formulated and proved for M_n, even without any differentiability assumption on M_n; in fact, Theorem 1 has a counterpart for locally compact metric spaces.

For Theorem 2 one requires a corresponding differentiability assumption for M_n. Since the proof mainly concerns a compact subset G that includes E in its interior, one can cover G by a finite number, N, of coordinate neighborhoods and then define $\psi_m(r)$ as in (2) so that the "max" applies both to $x \in K_r$ and to all of the N coordinate systems. The proof of Theorem 2 then carries over to this case, with each partial derivative being computed in one of the N coordinate systems. Similar remarks apply to Theorem 3.

In Theorems 1, 2, 3 we can replace the word "compact" by "closed." For, if E is unbounded, R_n can be imbedded by stereographic projection in the sphere S_n, and E becomes a set E_1 whose closure (E_1 plus "the point at infinity") is compact. The function F becomes a function of class $C^{(m)}$ in the complement of the closure of E_1. We can thus apply Theorem 1, 2, or 3, as generalized to the differentiable manifold S_n. The resulting representation of F as F_1/F_2 in S_n becomes the desired representation in R_n; in addition, $F_1(x)$ and $F_2(x)$ will now approach 0 as x approaches the point at infinity.

5. APPLICATION TO DIFFERENTIAL EQUATIONS

THEOREM 4. *Let* E *be a closed proper subset of* R_n. *Let a vector field* $f(x) = \{f_i(x_1, \cdots, x_n)\}$ *of class* $C^{(m)}$ ($m \in \{0, 1, 2, \cdots, \infty\}$) *be defined in* $D = R_n - E$. *Then there exist a vector field* g(x) *and a scalar* $\alpha(x)$, *defined and of class* $C^{(m)}$ *in* R_n, *such that* $\alpha(x) > 0$ *in* D, $\alpha(x) = 0$ *in* E, $g(x) = 0$ *in* E, *and* $g(x) = \alpha(x)\,f(x)$ *in* D.

In other words, the lengths of the vectors f(x) can be modified by multiplication by the positive scalar $\alpha(x)$ so that the modified vector field g(x) can be extended to E to remain of class $C^{(m)}$ and to become 0 on E. Thus the singularities of the vector field become *zeros* of the vector field.

Proof of Theorem 4. We apply Theorem 2 or Theorem 3 (with E a *closed* set, as in Section 4) to write $f_i(x) = f_{i1}(x)/f_{i2}(x)$, where $f_{i1}(x)$, $f_{i2}(x)$ are of class $C^{(m)}$

in R_n, $f_{i2}(x) > 0$ in D, and $f_{i1}(x) = f_{i2}(x) = 0$ on E. Then we set

$$\alpha(x) = \prod_{i=1}^{n} f_{i2}(x), \qquad g_i(x) = f_{i1}(x) \prod_{j=1}^{n}{}' f_{j2}(x)$$

where the ′ denotes that the factor for which $j = i$ is omitted in the product. We see at once that $\alpha(x)$ and $g(x) = \{g_i(x)\}$ have the properties asserted.

From Theorem 4 we obtain the desired conclusion for the differential equations (1). For example, let the f_i be of class $C^{(m)}$ in a deleted neighborhood D of a point x_0. We then set $E = R_n - D$ and apply Theorem 4. The new differential equations

$$\frac{dx_i}{d\tau} = g_i(x_1, \cdots, x_n) \qquad (i = 1, \cdots, n) \tag{7}$$

have right hand-members of class $C^{(m)}$ in all of R_n, so that the existence theorem is applicable at every point. The point x_0 is now an "equilibrium point" for equations (7): all g_i are 0 at x_0. The solutions of (7) have the same trajectories in D as the given equations (1), the difference being one of parametrization; along each solution

$$dt = \alpha(x_1, \cdots, x_n)\, d\tau \qquad (\alpha > 0). \tag{8}$$

For examples of such reparametrizations, we refer to [2, pp. 426-428 and 436-437].

6. APPLICATION TO EXTENSIONS OF FUNCTIONS

We give an example which, in turn, suggests generalizations. Let $F(x, y)$ be a function of two variables x and y, defined and continuous on a nonvoid subset D of the circle $K: x^2 + y^2 = 1$; let D be open relative to K. We then seek an extension ϕ of F to the interior of K. Such an extension ϕ can be obtained; it can even be analytic in x and y, in the interior.

In order to find ϕ, we regard K as a 1-dimensional differentiable manifold, as in Section 4, and apply Theorem 1, generalized, to obtain functions $F_1(x, y)$ and $F_2(x, y)$, continuous on K, such that $F = F_1/F_2$ in D and

$$F_2(x, y) > 0 \text{ in } D, \quad F_2(x, y) = 0 = F_1(x, y) \text{ in } E = K - D.$$

We then solve the Dirichlet problem for the unit circle with boundary values F_1, F_2 respectively; let the solutions be $\phi_1(x, y)$, $\phi_2(x, y)$. Now set $\phi(x, y) = \phi_1(x, y)/\phi_2(x, y)$. Since $F_2(x, y) \geq 0$ on K and $F_2(x, y) > 0$ in D, we can apply the maximum principle to conclude that $\phi_2(x, y) > 0$ in the interior of the circle. Thus $\phi(x, y)$ is an analytic function of x, y in the interior of K and is continuous in D plus the interior of K, with $\phi = F_1/F_2 = F$ in D. Accordingly, ϕ is the desired extension of F.

Remark. One can in fact choose $\phi(x, y)$ to be harmonic in the interior (see [1]).

REFERENCES

1. W. Kaplan, *Approximation by entire functions*, Michigan Math. J. 3 (1955), 43-52.

2. ———, *Ordinary differential equations*, Addison-Wesley, Reading, Mass., 1958.

3. H. Whitney, *Differentiable functions defined in closed sets, I*, Trans. Amer. Math. Soc. 36 (1934), 369-387.

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