ON A CONJECTURE OF GOODMAN CONCERNING MEROMORPHIC UNIVALENT FUNCTIONS

James A. Jenkins

1. Several authors have studied functions meromorphic and univalent in the unit circle, and some of them have made the further specialization of requiring the functions to possess a (simple) pole at a specified point of that circle [3, 4, 5, 6]. In addition, Goodman [1] has studied the class of functions that are *meromorphic and typically real* in the unit circle. Let such a function $f(z)$ have a pole at the point p $(0 < p < 1)$, and expansion about the origin given by

$$f(z) = z + \sum_{n=2}^{\infty} B_n z^n; \tag{1}$$

also, let

$$B(n, p) = (1 + p^2 + \cdots + p^{2n-2})/p^{n-1}.$$

It follows as a special case of a result of Goodman that

$$B_n \leq B(n, p).$$

Goodman denoted by $U(p)$ the class of functions $f(z)$, *meromorphic and univalent* in $|z| < 1$, with a pole at $z = p$ $(0 < p < 1)$ and with an expansion of the form (1) about the origin. In view of the preceding result, he made the conjecture (analogous to the Bieberbach conjecture) that the inequality $|B_n| \leq B(n, p)$ holds for each function in $U(p)$. Komatu [4] proved the inequality for the case $n = 2$.

The purpose of the present paper is to show that the conjecture is an easy consequence of the Bieberbach conjecture. In fact it is valid up to any stage for which the Bieberbach conjecture is true, so that in particular it holds at least for $n = 3$ and 4.

2. Let us denote by $E(p)$ the domain obtained from the unit circle $|z| < 1$ by deleting the segment $p \leq z < 1$, where $0 < p < 1$. Let us denote by $S(p)$ the class of functions regular and univalent in $E(p)$ with expansion of the form (1) about the origin. Then $S(1)$ is the class we usually denote by S [2, p. 1]. The domain $E(p)$ can be mapped conformally onto the unit circle by a function $k(z)$ with expansion about the origin

$$k(z) = (1 + p)^2 (4p)^{-1} z + \sum_{n=2}^{\infty} c_n z^n.$$

It is readily verified, for example from Löwner's parametric representation, see [7], that $c_n > 0$ for all n. Indeed, in equation (63), the function $\kappa(\tau) \equiv 1$. Now we

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see at once that if $g \in S$, then $4p(1+p)^{-2}gk \in S(p)$, and that conversely every function in $S(p)$ admits such a representation. Moreover if $g(z)$ is the function $z(1-z)^{-2}$, we find that $4p(1+p)^{-2}gk$ is the function

$$z[1-(p+p^{-1})z+z^2]^{-1}, \tag{2}$$

since it maps the unit circle onto the sphere with a slit along the negative real axis, the point p going into the point at infinity. This has about the origin the expansion

$$z + \sum_{n=2}^{\infty} B(n, p)z^n.$$

Finally, if $g(z)$ has the expansion about the origin

$$z + \sum_{n=2}^{\infty} A_n z^n,$$

then the function $4p(1+p)^{-2}gk$ has the expansion

$$z + \sum_{n=2}^{\infty} B_n z^n,$$

where B_n is a non-homogeneous linear combination of $A_2, \cdots, A_n$, that is, where

$$B_n = \sum_{j=2}^{n} \lambda_j^{(n)} A_j + \lambda_1^{(n)};$$

here the $\lambda_j^{(n)}$ $(j = 1, \cdots, n)$ are polynomials in c_k $(k = 2, \cdots, n)$ with non-negative coefficients, and thus they are themselves non-negative. Also, if in particular $A_j = j$ $(j = 1, 2, \cdots, n)$, then $B_n = B(n, p)$.

Now if the Bieberbach conjecture is valid up to index n, we have at once

$$|B_n| \leq \sum_{j=2}^{n} \lambda_j^{(n)} |A_j| + \lambda_1^{(n)} \leq \sum_{j=2}^{n} j\lambda_j^{(n)} + \lambda_1^{(n)} = B(n, p).$$

This completes the proof of the inequality stated in the following theorem.

THEOREM. *If* $f \in S(p)$ *and has the expansion* (1) *about the origin, and if the Bieberbach conjecture is valid up to index* N, *then*

$$|B_n| \leq B(n, p) \qquad (n = 2, \cdots, N). \tag{3}$$

In particular, this holds for $N = 4$. *Equality occurs in* (3) *only for the function* (2), *up to any index* N *such that to that point the Bieberbach conjecture admits equality only for the Koebe functions.*

The statement of equality is immediate.

Goodman's class U(p) is a subclass of S(p); thus the same result applies to it. It is clear that having a simple pole as a singularity at p plays no essential role in the result. Also, it is clear that having the singularity on the real axis is only a matter of normalization. Finally, it may be remarked that similar considerations apply to some of Goodman's results on typically real functions. In particular, at least part of his Theorem 7 in [1] admits substantial generalization.

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Washington University