THE LINDELÖF THEOREM AND THE REAL AND IMAGINARY PARTS OF NORMAL FUNCTIONS

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1. Let D and K denote respectively the open unit disk $|z| < 1$ and the unit circle $|z| = 1$ in the complex plane. A function f(z) defined in D is *normal* provided the family of functions

$$f_t(z) = f\left(\frac{z - t}{1 - \bar{t}z}\right) \qquad (t \in D)$$

is normal. If f(z) is a meromorphic function defined in D, we write $u(z) = \Re f(z)$ and $v(z) = \Im f(z)$. The cluster set, the outer angular cluster set, and the boundary cluster set of f at a point $e^{i\theta} \in K$ are denoted by $C(f, e^{i\theta})$, $C_A(f, e^{i\theta})$, $C_B(f, e^{i\theta})$, respectively, and $\mathscr{F}C(f, e^{i\theta})$ means the frontier of the set $C(f, e^{i\theta})$. If γ is an arc of $D \cup K$ terminating at the point $e^{i\theta} \in K$, then $C_\gamma(f, e^{i\theta})$ is the cluster set of f at $e^{i\theta}$ along γ; and, if E is a subset of $D \cup K - \{e^{i\theta}\}$ whose closure contains $e^{i\theta}$, then $C_E(f, e^{i\theta})$ is the cluster set of f at $e^{i\theta}$ on E.

Gehring and Lohwater [3, p. 165, Theorem 2] proved recently the following theorem:

Let f(z) *be holomorphic and bounded in* D, *and let* α *and* β *be arcs lying in* $D \cup K$ *and terminating at the point* $e^{i\theta} \in K$ *such that* f(z) *is continuous at all points of* $(\alpha \cup \beta) \cap K$ *except possibly at* $e^{i\theta}$. *If* $u(z) \to a$ *and* $v(z) \to b$ *as* $z \to e^{i\theta}$ *along* α *and* β, *respectively, then* f(z) *has the angular limit* $a + bi$ *at* $e^{i\theta}$.

This result is a generalization of a well-known theorem of Lindelöf, and the fact (established by Lehto and Virtanen [5, p. 53, Theorem 2]) that the Lindelöf theorem holds for normal meromorphic functions makes it natural to ask (as did Gehring in a conversation with the author) whether the Gehring-Lohwater theorem remains valid if the assumption that f(z) is holomorphic and bounded in D is replaced by the hypothesis that f(z) is meromorphic and normal in D. In Theorem 1 we answer this question in the negative; in Theorem 2 we show that the answer remains negative even if we assume that f(z) is holomorphic (and normal) instead of merely meromorphic. Then in Theorem 3 we obtain a generalization of the Gehring-Lohwater theorem to normal meromorphic functions under an additional assumption concerning a certain cluster set $C_{G(\alpha,\beta)}(f, e^{i\theta})$ and with a slightly modified conclusion, and we give examples to show the necessity of such an additional assumption as well as of the modification of the conclusion.

2. Noshiro [7, p. 154] has divided the class of normal meromorphic functions into two parts, obtaining what he calls functions of the first and second category, but what we prefer to call functions of the first and second kind in order to avoid any allusion to Baire category, which is not involved here. A normal meromorphic function is of the *first kind* if the family $\{f_t(z)\}$ $(t \in D)$ admits no constant limit; otherwise it is of the *second kind.* Noshiro has shown that normal meromorphic functions of the first kind in D possess some interesting properties, notably [7, p. 154, Theorem 5] the properties of having no asymptotic value and of assuming every value, including ∞, infinitely often in D.

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THEOREM 1. *There exists a normal meromorphic function* f(z) *of the first kind in* D *with the property that to every point* $e^{i\theta} \in K$ *there correspond arcs* α *and* β *in* D *terminating at* $e^{i\theta}$ *such that* $u(z) \equiv 0$ *on* α *and* $v(z) \equiv 0$ *on* β, *but* f(z) *has no angular limit at any point of* K.

Proof. Let f(z) be a Schwarzian triangle-function (see [6, Chapter II]) in D whose fundamental triangle has angles $\pi/2$, $\pi/7$, $\pi/3$, so that [6, pp. 125-126] f(z) is a normal meromorphic function in D, and of the first kind, as was noted by Noshiro [7, p. 156]; and let its system of triangles be that displayed in [4, p. 437, Fig. 122], where we assume that $f(0) = \infty$. Given any point $e^{i\theta} \in K$, it is then evident that there exists a sequence $z_0 = 0, z_1, z_2, \cdots$ of distinct vertices of the aforementioned triangles such that $z_n \to e^{i\theta}$ as $n \to \infty$, $f(z_n) = \infty$ $(n = 0, 1, 2, \cdots)$ and such that, for $n = 1, 2, 3, \cdots$, there exist triangles S_n, T_n of the figure satisfying the conditions that z_{n-1} is a vertex of S_n, z_n is a vertex of T_n, and the side s_n of S_n opposite z_{n-1} is identical with the side t_n of T_n opposite z_n. The value of f(z) at one of the end points of s_n is zero, and it follows from the mode of genesis of the function that there is an arc σ_n inside S_n and extending from z_{n-1} to the end point in question, such that f(z) is a pure imaginary number for every $z \in \sigma_n$, and hence $u(z) \equiv 0$ on σ_n. Similarly, there is an arc τ_n inside T_n and extending from the end point of $s_n \equiv t_n$ just considered to the point z_n, such that $u(z) \equiv 0$ on τ_n. Setting

$$\alpha = \bigcup_{n=1}^{\infty} (\sigma_n \cup \tau_n),$$

we have an arc α in D terminating at $e^{i\theta}$ such that $u(z) \equiv 0$ on α. On the sides of the triangles S_n, T_n $(n = 1, 2, 3, \cdots)$ the function f(z) is real, so that $v(z) \equiv 0$ there, and clearly there exists an arc β in D composed of sides of these triangles and terminating at $e^{i\theta}$ such that $v(z) \equiv 0$ on β. Finally, f(z), being a normal meromorphic function of the first kind, has no asymptotic value, as was noted above, and hence f(z) has no angular limit at any point of K.

3. It is not possible to obtain a normal *holomorphic* function with the properties possessed by the function f(z) in Theorem 1, because every normal holomorphic function in D has angular limits at the points of an everywhere dense subset of K [1, Corollary 1]. Nevertheless, the next theorem shows that a normal holomorphic function in D can thwart the Gehring-Lohwater conclusion at every point of K (although for some points of K our continuity assumption is weaker than that in the Gehring-Lohwater theorem).

THEOREM 2. *There exists a normal holomorphic function* f(z) *in* D *and an enumerable subset* E *of* K *with the properties that to every point* $e^{i\theta} \in K$ *there correspond arcs* α *and* β *in* $D \cup K$ *terminating at* $e^{i\theta}$ *such that* f(z) *is continuous (in the extended sense) on* $\alpha \cup \beta$, *except possibly at* $e^{i\theta}$, *and, if* $e^{i\theta} \in K - E$, *then* $u(z) \equiv 0$ *on* α, $v(z) \equiv 0$ *on* β, *but* f(z) *does not have an angular limit at* $e^{i\theta}$; *whereas if* $e^{i\theta} \in E$, *there correspond to* $e^{i\theta}$ *real numbers* a, b *such that* $u(z) \equiv a$ *on* α, $v(z) \equiv b$ *on* β, *but* f(z) *does not have the angular limit* $a + bi$ *at* $e^{i\theta}$.

Proof. Let f(z) be the modular function in D; it is holomorphic there and omits the values 0, 1, ∞, so that it is also normal. Denote by E the enumerable set of vertices of the modular figure [4, p. 432, Fig. 119]. To each vertex there corresponds, in the course of construction of the function, one of the values 0, 1, ∞, and the vertex will be dubbed a 0-, 1-, or ∞-vertex, accordingly. Each triangle of the modular figure has a 0-, a 1-, and an ∞-vertex, and its sides will be called the 01-, the ∞0-, and the ∞1-side after the two vertices that the side in question joins.

We consider first the points of E. Let $e^{i\theta}$ be an ∞-vertex, and let T be a triangle of the modular figure of which $e^{i\theta}$ is a vertex. Then there is an arc α in T extending from the 0-vertex of T to the point $e^{i\theta}$, such that $u(z) \equiv 0$ on α. If we denote by β the $\infty 0$-side of T, then $v(z) \equiv 0$ on β. The angular limit of f(z) at $e^{i\theta}$, however, is ∞, not 0. Now let $e^{i\theta}$ be a 0-vertex. Then there is a sequence of "consecutive" $\infty 1$-sides of triangles of the modular figure approaching the point $e^{i\theta}$, whose union is an arc β in $D \cup K$ terminating at $e^{i\theta}$ on which f(z) is continuous (except at $e^{i\theta}$) and $v(z) \equiv 0$ on β. The successive vertices of the $\infty 1$-sides of the sequence can be joined by arcs on which $u(z) \equiv 1$, such that the union of these arcs is an arc α in $D \cup K$ terminating at $e^{i\theta}$ on which f(z) is continuous. The angular limit of f(z) at $e^{i\theta}$, however, is 0, not 1. Finally, let $e^{i\theta}$ be a 1-vertex. Then there is a sequence of consecutive $\infty 0$-sides approaching the point $e^{i\theta}$, whose union is an arc β terminating at $e^{i\theta}$ on which $v(z) \equiv 0$. The successive vertices of the $\infty 0$-sides can be joined by arcs on which $u(z) \equiv 0$, such that the union of these arcs is an arc α terminating at $e^{i\theta}$. The angular limit of f(z) at $e^{i\theta}$, however, is 1, not 0.

We turn now to the points of K - E. Let T be a triangle of the modular figure that does not contain the origin in its interior, and let s be one of its sides. We say that s *spans the arc* A on K if A is the minor subarc of K having the same end points as s but containing no other vertex of T. Accordingly, only two sides of T span arcs on K, and the arc spanned is uniquely determined by the side; the closure of the region bounded by the side and the arc that it spans will be called the domain G spanned by the arc. Suppose that $e^{i\theta} \in K - E$, that the side s of the triangle T spans the arc A, and that A contains $e^{i\theta}$. Since s is either an $\infty 1$-side, an $\infty 0$-side, or a 01-side, it contains either an ∞-vertex or a 0-vertex (possibly both). The midpoint of A divides A into two subarcs; denote one which contains $e^{i\theta}$ by A'. We shall show, below, that it is possible to join the ∞- or 0-vertex of s to an ∞- or a 0-vertex P_1 of the modular figure on A' by means of a finite number of consecutive $\infty 0$-sides of the modular figure, all of which lie in the domain G spanned by s. The union of these sides is an arc β_1 in G along which f(z) is continuous and real, so that $v(z) \equiv 0$ on β_1. Moreover, the ∞- and 0-vertices of the consecutive $\infty 0$-sides can be joined by arcs in triangles having these sides and lying in G, such that f(z) is a pure imaginary at every point of these arcs. Their union α_1, then, is an arc in G terminating at the terminus, P_1, of β_1; on α_1, $u(z) \equiv 0$ and f(z) is continuous. It will be evident from the construction of β_1 that $e^{i\theta}$ lies on a proper subarc of A that is spanned by a side of a triangle of the modular figure, where one vertex of that side is P_1. The construction process can then be repeated, and this leads successively to arcs $\beta_2, \alpha_2, \beta_3, \alpha_3, \cdots$ such that, setting

$$\alpha = \bigcup_{n=1}^{\infty} \alpha_n, \qquad \beta = \bigcup_{n=1}^{\infty} \beta_n,$$

we have arcs α and β in $D \cup K$ terminating at $e^{i\theta}$ such that f(z) is continuous on $\alpha \cup \beta$ except at $e^{i\theta}$, $u(z) \equiv 0$ on α, and $v(z) \equiv 0$ on β. As is well known, f(z) does not have an angular limit at $e^{i\theta}$.

Suppose first that s is an $\infty 1$-side. Reflect T in s, getting a triangle T' with its 0-vertex at an interior point of A. If A' has the ∞-vertex of T as one of its end points, reflect T' in its $\infty 0$-side, the resulting triangle in its $\infty 1$-side, the ensuing triangle in its $\infty 0$-side, and so on, until a triangle is obtained whose $\infty 0$-side has its 0-vertex on A' and whose 01-side spans an arc containing $e^{i\theta}$; the $\infty 0$-side of this triangle is taken to be β_1. If, however, A' has the 1-vertex of T as one of its end points, reflect T' in its 01-side, the resulting triangle in its ∞1-side, the ensuing triangle in its 01-side, and so forth, until a triangle is obtained whose ∞0-side has a vertex on A' and spans an arc containing $e^{i\theta}$; the union of this last ∞0-side, the ∞0-side of T', and the ∞0-sides of all the other triangles that were obtained by reflections, is taken to be β_1.

Suppose next that s is a 01-side. Interchange "∞" and "0" in the preceding paragraph.

Suppose finally that s is an ∞0-side. If A' has the ∞-vertex of T as one of its end points, then an argument like that in the first half of the penultimate paragraph yields a triangle whose ∞0-side has its 0-vertex on A' and whose ∞-vertex is the ∞-vertex of T; this ∞0-side or, if necessary, its union with the ∞0-side of T, is taken to be β_1. If, however, A' has the 0-vertex of T as one of its endpoints, then an argument like that in the first half of the last paragraph affords a triangle whose ∞0-side has its ∞-vertex on A' and whose 0-vertex is the 0-vertex of T; this ∞0-side, or, if need be, its union with the ∞0-side of T, is taken to be β_1.

4. We now introduce the notion of a *quadrantal set*, which will enable us to state the next theorem more concisely. Let a and b be finite real numbers. The straight lines $\Re z = a$, $\Im z = b$ divide the plane into four quadrants, which we consider as open sets, and to which we refer as the *quadrants at* a + bi; they are numbered in the usual manner. We say that a set S is *quadrantal* at a + bi if S contains a point belonging to some quadrant at a + bi and, whenever S contains a point of some quadrant at a + bi, S contains every point of that quadrant.

If α and β are arcs lying in $D \cup K$ and terminating at the point $e^{i\theta} \in K$, there is no loss of generality, for our purposes, in assuming that α and β have a common initial point in D. We define the set $G(\alpha, \beta)$ to consist of all points of α and β except $e^{i\theta}$, as well as the points of every subregion of D whose frontier is a subset of $\alpha \cup \beta$.

THEOREM 3. *Let* f(z) *be a normal meromorphic function in* D, *and let* α *and* β *be arcs lying in* $D \cup K$, *terminating at the point* $e^{i\theta} \in K$, *and such that* f(z) *is continuous at all points of* $(\alpha \cup \beta) \cap K$ *except possibly at* $e^{i\theta}$. *If* $u(z) \to a$ *and* $v(z) \to b$ *as* $z \to e^{i\theta}$ *along* α *and* β, *respectively, and if* $C_{G(\alpha,\beta)}(f, e^{i\theta})$ *is not quadrantal at* a + bi, *then* f(z) *has the angular limit* a + bi *or* ∞ *at* $e^{i\theta}$.

Proof. Suppose first that at least one of the values a, b, say a, is infinite, so that $a + bi = \infty$. Then $f(z) \to \infty$ as $z \to e^{i\theta}$ along α, which implies [5, p. 53, Theorem 2] that f(z) has the angular limit ∞ at $e^{i\theta}$, and we thus obtain the conclusion of our theorem without the aid of any assumption concerning $C_{G(\alpha,\beta)}(f, e^{i\theta})$.

Suppose, however, that both a and b are finite real numbers. We modify the proof [3, pp. 166-167] of the Gehring-Lohwater theorem. Since, by hypothesis, f(z) is continuous at all points of $(\alpha \cup \beta) \cap K$ except possibly at $e^{i\theta}$, we can replace α and β, if $(\alpha \cup \beta) \cap K - \{e^{i\theta}\}$ is not empty, by arcs α' and β' lying in D and terminating at $e^{i\theta}$ such that $u(z) \to a$ and $v(z) \to b$ as $z \to e^{i\theta}$ along α' and β', respectively, and such that

$$C_{G(\alpha,\beta)}(f, e^{i\theta}) = C_{G(\alpha',\beta')}(f, e^{i\theta});$$

we may therefore assume, without loss of generality, that $(\alpha \cup \beta) - \{e^{i\theta}\} \subset D$. Next we construct a Jordan curve γ such that $e^{i\theta} \in \gamma$, $\gamma - \{e^{i\theta}\} \subset D$, $G(\alpha, \beta)$ lies in the interior, J, of γ,

(1) $$C_J(f, e^{i\theta}) = C_{G(\alpha, \beta)}(f, e^{i\theta}),$$

and

(2) $$C_\gamma(f, e^{i\theta}) \subseteq C_\alpha(f, e^{i\theta}) \cup C_\beta(f, e^{i\theta}).$$

Let $z = z(\zeta)$ map the open unit disk in the ζ-plane conformally onto J, let $e^{i\theta_1}$, α_1, β_1 denote the preimages of $e^{i\theta}$, α, β, respectively, and let $w = f_1(\zeta) = f(z(\zeta))$. It follows from (2) that

$$C_B(f_1, e^{i\theta_1}) \subseteq C_\alpha(f, e^{i\theta}) \cup C_\beta(f, e^{i\theta}),$$

and hence in the w-plane the set $C_B(f_1, e^{i\theta_1})$ is a subset of the union X of the lines $\Re w = a$, $\Im w = b$, and the point ∞. We wish to derive the equality

(3) $$C(f_1, e^{i\theta_1}) = C_B(f_1, e^{i\theta_1})$$

from the familiar inclusions

(4) $$\mathscr{F}C(f_1, e^{i\theta_1}) \subseteq C_B(f_1, e^{i\theta_1}) \subseteq C(f_1, e^{i\theta_1}),$$

and to this end it suffices to show that

(5) $$C(f_1, e^{i\theta_1}) \subseteq \mathscr{F}C(f_1, e^{i\theta_1}).$$

Now a consequence of (1) is that $C(f_1, e^{i\theta_1})$ is not quadrantal at $a + bi$. This means that either $C(f_1, e^{i\theta_1})$ contains no point of any quadrant at $a + bi$, or $C(f_1, e^{i\theta_1})$ contains a point, but not every point, of some quadrant Q at $a + bi$. In the first case it is clear that $C(f_1, e^{i\theta_1})$ can have no interior point, and hence (5) holds. In the second case, suppose that $C(f_1, e^{i\theta_1})$ contains the point $w_1 \in Q$ but not the point $w_2 \in Q$. Then $\mathscr{F}C(f_1, e^{i\theta_1})$ contains a point w_0 on the segment bounded by w_1 and w_2, and evidently w_0 does not belong to X. This, however, contradicts (4) and the fact noted above that

(6) $$C_B(f_1, e^{i\theta_1}) \subseteq X.$$

The second case therefore cannot occur, and thus the validity of (3) is established. It follows from (3) and (6) that $C(f_1, e^{i\theta_1})$ is nowhere dense in the w-plane, and Gross's theorem (see [3, p. 166]) implies that

$$C_A(f_1, e^{i\theta_1}) \subseteq C_{\alpha_1}(f_1, e^{i\theta_1}) \cap C_{\beta_1}(f_1, e^{i\theta_1}) \subseteq \{a + bi, \infty\}.$$

Thus $f_1(\zeta)$ tends to $a + bi$ or to ∞ as $\zeta \to e^{i\theta_1}$ in some Stolz angle at $e^{i\theta_1}$, and hence $f(z)$ tends to $a + bi$ or to ∞ as $z \to e^{i\theta}$ along some arc in D terminating at $e^{i\theta}$, and the conclusion that $f(z)$ has either the angular limit $a + bi$ or the angular limit ∞ at $e^{i\theta}$ now follows from [5, p. 53, Theorem 2].

5. In this section we illustrate Theorem 3 by means of a few examples whose details can be readily supplied by the reader.

(a) A closer study of the figure of the system of triangles associated with the function considered in the proof of Theorem 1 shows that there exists a normal meromorphic function in D with the property that if S is either the closure of one quadrant at the point 0, or the closure of two adjacent quadrants, or the closure of two alternate quadrants, or the closure of three quadrants, or the whole complex plane, then there exist a point $e^{i\theta} \in K$ and suitable arcs α, β in D terminating at $e^{i\theta}$, such that

$$C_{G(\alpha,\beta)}(f, e^{i\theta}) = S, \qquad u(z) \equiv 0 \text{ on } \alpha, \qquad v(z) \equiv 0 \text{ on } \beta,$$

and yet f(z) has no angular limit at $e^{i\theta}$. This indicates that the notion of quadrantal set as we have defined it is germane to the situation treated in Theorem 3.

(b) The need for including ∞ in the conclusion of Theorem 3 is shown by the function w = f(z) that maps D conformally onto the first quadrant of the w-plane in such a manner that z = 1 goes into $w = \infty$ and z = -1 goes into w = 0: take α to be the upper half and β to be the lower half of K, $e^{i\theta}$ to be the point 1, and a = b = 0.

(c) We adduce a function f(z), cited by Carathéodory [2, p. 271], to show that if $C_{G(\alpha,\beta)}(f, e^{i\theta})$ *is* quadrantal at a + bi then f(z) can have an angular limit at $e^{i\theta}$ different from a + bi, even if f(z) is holomorphic in D and omits there an entire half-plane of values. The function alluded to is

$$w = f(z) = i\,\frac{\exp\frac{1+z}{1-z} + 1}{\exp\frac{1+z}{1-z} - 1}.$$

It is holomorphic in D, and omits there all values belonging to the lower half-plane, so that it is normal in D. At the point $1 \in K$ it has the angular limit i. Along the radius α of K terminating at the point 1, $u(z) \equiv 0$, whereas along the lower half β of K, $v(z) \equiv 0$, and f(z) is continuous at every point of $\beta - \{1\}$. In this case, $C_{G(\alpha,\beta)}(f, 1)$ is the closure of the upper half-plane.

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