THE TOPOLOGICAL STRUCTURE OF TRAJECTORIES

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1. INTRODUCTION

Let V_n be a connected differentiable manifold of class C^r $(r \geq 1)$, and let $X(x, t)$ $(x \in V_n, t \in E_1)$ define a field of tangent vectors on $V_n \times E_1$. We shall always assume that $X(x, t)$ is of class C^r $(r \geq 1)$. Thus we have a system of equations,

$$\frac{dx}{dt} = X(x, t).$$

If $x(t)$ is a solution of these equations, we call the curve $(x(t), t) \subset V_n \times E_1$ a *trajectory*. We consider only maximal trajectories; that is, the function $x(t)$ is defined for $a < t < b$, where the interval (a, b) is maximal (a or b may be infinite).

Lefschetz has defined the general solution of a system of equations [1, p. 39]. We shall give a second definition and analyze the relation of these general solutions to the topological structure of the space of trajectories.

2. DIFFERENTIAL SYSTEMS WHICH ADMIT A SECTION

Definition 2.1. A *section* for a differential system on $V_n \times E_1$ is a manifold V_n', together with a homeomorphism f of V_n' into $V_n \times E_1$, with the property that $f(V_n')$ meets each trajectory exactly once.

THEOREM 2.2. *Assume we are given a differential system defined by a function* $X(x, t)$ *on* $V_n \times E_1$. *The trajectories of this system form a decomposition of* $V_n \times E_1$ *into closed sets. Let* V *be the space obtained by the identification of each trajectory to a single point. If* V *is a Hausdorff space, then* V *is an* n-*dimensional manifold. If* $X(x, t)$ *and* V_n *are of class* C^m, *then* V *is of class* C^m $(m \geq 1)$.

Proof. Let $p: V_n \times E_1 \to V$ be the mapping induced by the identification. Let $\{N_i\}$ be a countable collection of n-cells which form a basis for the topology of V_n, and $\{I_j\}$ a collection of 1-cells which form a basis for E_1. Then the collection $\{N_i \times I_j\}$ is a basis for $V_n \times E_1$. Choose $t_j \in I_j$ and let $f_{ij}: N_i \to N_i \times t_j$ be defined by $f_{ij}(x) = (x, t_j)$. Let $g_{ij}: N_i \to V$ be defined by $g_{ij} = p(f_{ij}(x))$, and define $U_{ij} = g_{ij}(N_i)$. Then $p^{-1}(U_{ij})$ is the set of all points which lie on a trajectory passing through $N_i \times t_j$. Thus $p^{-1}(U_{ij})$ is open in $V_n \times E_1$, and U_{ij} is open in V [1, p. 52]. If U is an open set in V and $y \in U$, then there exists a cell $N_i \times I_j$, contained in $p^{-1}(U)$, with the property that $p^{-1}(y)$ meets $N_i \times t_j$. Then $y \in U_{ij}$ and $U_{ij} \subset U$. Thus, the collection $\{U_{ij}\}$ is a basis for the topology of V. The mappings g_{ij} are one-to-one, and the g_{ij}^{-1} are continuous, since the g_{ij} are open.

Now assume that $X(x, t)$ is of class C^m. Then the solution $x(t, x_0, t_0)$, where $x(t_0, x_0, t_0) = x_0$, is of class C^m in (x_0, t_0). We must show that

$$g_{pq}^{-1} g_{ij}: g_{ij}^{-1}(U_{ij} \cap U_{pq}) \to g_{pq}^{-1}(U_{ij} \cap U_{pq})$$

Received October 7, 1959.

is of class C^m. The mappings $f_{ij}: N_{ij} \to V_n \times E_1$ are of class C^m. If

$$z \in g_{ij}^{-1}(U_{ij} \cap U_{pq}),$$

then

$$f_{ij}(z) = (z, t_j) \quad \text{and} \quad g_{pq}^{-1} g_{ij}(z) = f_{pq}^{-1}(x(t_q, z, t_j), t_q).$$

Thus $g_{pq}^{-1} g_{ij}(z)$ is of class C^m in z, since f_{pq}^{-1} and $x(t, x_0, t_0)$ are of class C^m. Therefore, if V is a Hausdorff space, then it is a manifold of class C^m.

THEOREM 2.3. *If a differential system on* $V_n \times E_1$ *admits a section by* V_n', *then there exists a homeomorphism of* $V_n' \times E_1$ *onto* $V_n \times E_1$ *which maps the lines* $y \times E_1$, *where* $y \in V_n'$, *onto trajectories.*

Proof. Since V_n is an n-manifold of class C^1, there exists an imbedding, of class C^1, of $V_n \times E_1$ into E_{2n+3} which is without limit set [2, p. 113]. Using the metric in E_{2n+3}, we can define arc length along the trajectories $(x(t), t)$ in $V_n \times E_1$. Let $F: V_n \times E_1 \to E_{2n+3}$ be the imbedding. Given the trajectory $(x(t), t)$, where $a < t < b$, let $s(t)$ be the arc length along this trajectory, measured from the intersection with $f(V_n')$. Here the mapping $f: V_n' \to V_n \times E_1$ is that which defines the section. Then, $\lim_{t \to b-} s(t) = \infty$ and $\lim_{t \to a+} s(t) = -\infty$. For if not, then there exists a sequence $t_i \to b$ such that the set $(x(t_i), t_i)$ has no limit point and

$$F(x(t_i), t_i) \to y_0 \in E_{2n+3}.$$

But this is impossible, since the imbedding is without limit set.

Define a mapping $G: V_n' \times E_1 \to V_n \times E_1$ by defining $G(y, s)$ to be the point on the trajectory through $f(y)$ whose distance from $f(y)$, measured by the arc length along the trajectory, is equal to s. Then G is the desired homeomorphism.

THEOREM 2.4. *Let* $X(x, t)$ *define a differential system on* $V_n \times E_1$, *where* $X(x, t)$ *is of class* C^1. *Then the differential system admits a section if and only if the identification space* V, *obtained by identifying trajectories to a point, is a Hausdorff space. Also, if* $f: V_n' \to V_n \times E_1$ *is a section for the system, then* V_n' *is homeomorphic to* V.

Proof. Assume that V is a Hausdorff space. Let $p: V_n \times E_1 \to V$ be the projection defined by the identification. Let $N_i \times t_i$ (where N_i is an n-cell in V_n, $t_i \in E_i$) be a collection of n-cells in $V_n \times E_1$ such that the collection $p(N_i \times t_i)$ is a basis for V. Let $U_i = p(N_i \times t_i)$. We now imbed $V_n \times E_1$ into E_{2n+3}, as in the proof of Theorem 2.3. Then $p^{-1}(U_i)$ is homeomorphic to $N_i \times E_1$, by the argument used in Theorem 2.3. By Theorem 2.2, V is a manifold of class C^1; therefore V can be triangulated [2, p. 124]. Thus, $V_n \times E_1$ is a fibre space over V with fibre E_1. Since E_1 is contractible, there exists a cross section $f: V \to V_n \times E_1$ such that $pf(x) = x$ for all $x \in V$. Thus $V_n \times E_1$ admits a section by V. The existence of the cross section is proved by first subdividing the triangulation in such a way that each simplex σ^n of the triangulation is contained in some one of the cells U_i. Then $p^{-1}(\sigma^n)$ is homeomorphic to $\sigma^n \times E_1$. Let K^r be the r-skeleton of the triangulation; that is, let K^r be the union of all r-simplexes in the triangulation. We may define a map $f^0: K^0 \to V_n \times E_1$ in such a way that $pf^0(x) = x$ for all $x \in K^0$. Assume we have a mapping $f^r: K^r \to V_n \times E_1$ with the property that $pf^r(x) = x$ for all $x \in K^r$. Since $p^{-1}(\sigma^{r+1})$ is homeomorphic to $\sigma^{r+1} \times E_1$, we can extend f^r to $f^{r+1}: K^{r+1} \to V_n \times E_1$ in such a way that $pf^{r+1}(x) = x$. Then $f^n: K^n = V \to V_n \times E_1$ is the desired section.

If a differential system on $V_n \times E_1$ admits a section by V_n', then V is homeomorphic to V_n'. This fact is an easy consequence of Theorem 2.3. Thus V must be a Hausdorff space, since V_n' is a Hausdorff space.

Example. Let a family of trajectories in $E_1 \times E_1$ be given by $(x(t), t)$, where:

a) $x(t) = C \quad (C \leq 0)$, b) $x(t) = \frac{1}{t^2 + C} \quad (0 < C < \infty)$, c) $x(t) = C + \frac{1}{t^2} \quad (0 \leq C < \infty)$.

The space obtained by identifying each trajectory to a point is not a Hausdorff space. This family of trajectories does not arise from a differential system of class C^1, but a modification can be found which is of class C^1.

3. GENERAL SOLUTIONS

The definition below is similar to that given by Lefschetz [1, p. 39].

Definition 3.1. Assume that we are given a differential system on $V_n \times E_1$. A *general solution* for this differential system is an n-cell N, together with a continuous mapping F of $N \times E_1$ onto $V_n \times E_1$, with the property that F maps $y \times E_1$ onto a trajectory, for each point y in N. We do not require that F be one-to-one.

THEOREM 3.2. *Each differential system has a general solution.*

Proof. Choose a countable collection of n-cells $\{N_i\}$ contained in V_n, and a collection of points $t_i \in E_1$ such that 1) $\overline{N}_i$ is a closed n-cell, 2) if P_i is the union of all points which lie on a trajectory through $N_i \times t_i$, then $\bigcup_1^\infty P_i = V_n \times E_1$. Let Q_i be the interior of an n-cube of width 1/4 centered about the point $(i, 0, \cdots, 0) \in E_n$. Let $\overline{R}_i$ be a closed n-cell in E_n (a "strip") joining Q_i to Q_{i+1}. Choose $\overline{R}_i$ in such a way that if

$$N = \left(\bigcup_1^\infty Q_i\right) \cup \left(\bigcup_1^\infty \overline{R}_i\right),$$

then N is an n-cell. Let $\overline{f}_i$ be a homeomorphism of $\overline{Q}_i$ onto $\overline{N}_i$. If f_i is the restriction of $\overline{f}_i$ to Q_i, then f_i is a homeomorphism of Q_i onto N_i. The mappings $\overline{f}_i$ and $\overline{f}_{i+1}$ are defined on $\overline{R}_i \cap \overline{Q}_i$ and $\overline{R}_i \cap \overline{Q}_{i+1}$, respectively. Extend these mappings to a mapping $g_i: R_i \to V_n \times E_1$. Then the mappings $\{f_i\}$ and $\{g_i\}$ induce a mapping $f: N \to V_n \times E_1$. We can define arc length along the trajectories in $V_n \times E_1$, by the same method as in Theorem 2.3. Define a mapping $G: N \times E_1 \to V_n \times E_1$ by defining $G(y, s)$ to be that point on the trajectory through $f(y)$ whose distance from $f(y)$, measured by the arc length along the trajectory, is equal to s. Here the mapping f is as defined above, and $y \in N$.

REFERENCES

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