SOME REMARKS ON SET THEORY, VIII

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This paper discusses some problems similar to questions considered in earlier communications of the same title [2], [3] and to some questions treated by P. Erdös and R. Rado [4], [5].

1. ON INDEPENDENT SETS

Let M be a set (in this note, M will denote the set of real numbers), and to each $x \in M$, let there correspond a set $S(x) \subset M$, called the *picture* of s, such that $x \notin S(x)$. A subset M' of M is called *independent* (or *free*) if, for each pair of points x and y in M, $x \notin S(y)$ and $y \notin S(x)$. In [2, I, p. 52] it was conjectured that if M is the set of real numbers, and if the measure of S(x) is bounded and S(x) is not everywhere dense, then there always exists an independent pair. In fact, it is easy to see that if we assume $\mathfrak{c} = \aleph_1$, then this conjecture is false. To construct a counterexample, we well-order M into an Ω_1-sequence $\{x_\alpha\}$ $(\alpha < \Omega_1)$. For each α, we write

$$S(x_\alpha) = S_1(x_\alpha) \cup S_2(x_\alpha),$$

where $S_1(x_\alpha)$ is the interval $(x_\alpha, x_\alpha + 1)$, and where $x_\beta \in S_2(x_\alpha)$ provided $\beta < \alpha$ and x_β does not lie in the interval $(x_\alpha - 1, x_\alpha)$. Clearly, $S(x_\alpha)$ has measure 1 (the set $S_2(x_\alpha)$ is at most denumerable) and is not everywhere dense, and no two points are independent.

Instead of the hypothesis that $\mathfrak{c} = \aleph_1$, we have used only the hypothesis that the measure of every set of power less than $\mathfrak{c}$ is 0. In fact, we need only the hypothesis that the set of real numbers can be well-ordered into a sequence $\{x_\alpha\}$ $(\alpha < \Omega_\mathfrak{c})$ such that every set which is not cofinal with $\Omega_\mathfrak{c}$ has measure 0. Denote this hypothesis by H_0. We do not know whether H_0 is equivalent to the hypothesis that each set of power less than $\mathfrak{c}$ has measure 0. Further, we do not know whether, if S(x) has the properties above, the negation of H_0 implies the existence of an independent pair.

Piranian (private communication) recently asked what can be said about independent points if each S(x) has measure 0 and is not everywhere dense.

THEOREM 1. *If* S(x) *has measure* 0 *and is not everywhere dense, there exists an independent pair; under the additional assumption* H_0, *an independent triplet need not exist.*

Proof. Let $A = \{a_n\}$ $(1 \leq n < \infty)$ be a denumerable dense set. Then $\bigcup_{n=1}^{\infty} S(a_n)$ is clearly of measure 0, and its complement contains a point b. Since S(b) is not everywhere dense, there exists an m such that $a_m \notin S(b)$. Clearly, a_m and b are independent.

On the other hand, let $\{x_\alpha\}$ $(\alpha < \Omega_\mathfrak{c})$ be a well-ordering of M. For $0 < \alpha < \Omega_\mathfrak{c}$, let $S(x_\alpha)$ be the set of those x_β $(\beta < \alpha)$ that have the same sign as x_α (here the sign

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of 0 is taken to be positive). Then $S(x_\alpha)$ is not everywhere dense; also, under the hypothesis H_0, it has measure 0. Clearly there is no independent triplet; this completes the proof of Theorem 1. We are unable to decide about the existence of an independent triplet, under the assumption that H_0 is false.

Theorem 1 can easily be strengthened: If each $S(x)$ has measure 0 and is nowhere dense, then there exist sets A and B, of power $\aleph_0$ and $\mathfrak{c}$, respectively, such that every pair x, y with $x \in A$ and $y \in B$ is independent. We can not decide whether the sets A and B can be chosen so that both have power $\mathfrak{c}$.

THEOREM 2. *If each picture* $S(x)$ *is bounded and has outer measure at most* 1, *then for every positive integer* k *there exists a set of* k *independent points.*

In the proof, we shall use the following well-known lemma: Let I be a bounded set, and let $\{B_n\}$ $(1 \leq n < \infty)$ be a sequence of subsets of I, each of inner measure greater than a fixed positive constant. Then there exists an infinite sequence $\{n_j\}$ such that $\bigcap_{j=1}^{\infty} B_{n_j}$ is not empty.

Instead of the conclusion in Theorem 2, we shall prove the following slightly stronger result, by induction on k: For each n, there exists an independent k-tuplet $\{u_i^{(n)}\}_{i=1}^{k}$ satisfying the condition $n < u_1^{(n)} < u_2^{(n)} < \cdots < u_k^{(n)}$. For $k = 1$, each $u_1^{(1)} > n$ satisfies the requirement, since by definition each point constitutes an independent set. Assume that we have demonstrated the existence of an independent $(k - 1)$-tuplet whose elements are arbitrarily large. Let I_{nk} denote the interval $(n, n + k)$. Corresponding to each integer m, there exists an independent $(k - 1)$-tuplet $\{u_i^{(m)}\}_{i=1}^{k-1}$ with $m < u_1^{(m)} < \cdots < u_{k-1}^{(m)}$. Since the outer measure of $\bigcup_{i=1}^{k-1} S(u_i^{(m)})$ is at most $k - 1$, there exists a set B_m, of inner measure at least 1, which lies in I_{nk} and does not meet $\bigcup_{i=1}^{k-1} S(u_i^{(m)})$. By our lemma, there exists an increasing sequence $\{m_j\}$ and a point x in I_{nk} such that $x \notin \bigcup_{j=1}^{\infty} \bigcup_{i=1}^{k-1} S(u_i^{(m_j)})$. Since the set $S(x)$ is bounded, it does not meet the set $\{u_i^{(m_j)}\}_{i=1}^{k-1}$, if j is large enough. In other words, if we write

$$u_1^{(n)} = x, \quad u_2^{(n)} = u_1^{(m_j)}, \quad \cdots, \quad u_k^{(n)} = u_{k-1}^{(m_j)},$$

the set $\{u_i^{(n)}\}_{i=1}^{k}$ is independent, and our proof is complete.

We can not decide whether the hypothesis of Theorem 2 implies the existence of an infinite independent set. A nondenumerable independent set clearly need not exist; to see this, let $S(x)$ consist of the two intervals $(x - 1/2, x)$ and $(x, x + 1/2)$.

If we replace the hypothesis that $S(x)$ is bounded by the hypothesis that $S(x)$ is closed, the existence of an independent set of cardinality $\mathfrak{c}$ follows from a theorem of Fodor [6]. But under the hypothesis that $S(x)$ has outer measure at most 1 and that the set $\{x\} \cup S(x)$ is closed, we have not been able to prove the existence even of an independent pair.

We again call attention to two problems mentioned in earlier papers. In [2, I, p. 53], it was shown that if each picture $S(x)$ is nowhere dense, then there exists an infinite independent set. Does there exist an uncountable infinite set? We can not even answer the following simpler question: Does there exist an uncountable independent set, if none of the pictures $S(x)$ contains a subset of type η (or if each picture $S(x)$ is a sequence of type ω with the only limit point x)? Let $\{E_\alpha\}$ $(1 < \alpha < \Omega_{\mathfrak{c}})$ be a family of $\mathfrak{c}$ sets of positive measure. Can it happen that each subfamily of power $\aleph_1$ of the sets E_α has an empty intersection? In [2, II, p. 173], it was pointed out that the problem is obvious if $\mathfrak{c} = \aleph_1$.

2. ON GRAPHS WHOSE VERTICES ARE REAL NUMBERS

In a graph G, a set S of vertices is *independent* if no two vertices in S are connected by an edge. A subgraph G' of G is a *complete graph* if each pair of its vertices is connected by an edge in G'. We denote by G_M a graph whose vertices are the elements of M, the set of real numbers. It was proved by Dushnik and Miller [1, Theorem 5.22] that if $\mathfrak{m}$ is a transfinite cardinal, then every graph of power $\mathfrak{m}$ contains either an infinite complete subgraph or an independent set of vertices whose power is $\mathfrak{m}$; in the notation of [4], this statement takes the form $\mathfrak{m} \rightarrow (\mathfrak{m}, \aleph_0)^2$. We now assume the continuum hypothesis and reach a slightly stronger conclusion.

THEOREM 3. *If* $\mathfrak{c} = \aleph_1$, *then each graph* G_M *contains either an infinite complete subgraph or an independent set of vertices of positive outer measure.*

It would be easy to give a direct proof of Theorem 3; but the theorem follows more quickly from the well-known result of Sierpiński [8, p. 31] that if $\mathfrak{c} = \aleph_1$, then there exists a set $S \subset M$, of power $\mathfrak{c}$, which meets every set of measure 0 in a set which is at most denumerable. Let G_M be any graph whose vertices constitute the set M, and let G_S denote the subgraph of G_M which is determined by Sierpiński's set S. By the theorem of Dushnik and Miller, G_S has either an infinite complete subgraph or an independent set S' of vertices whose power is $\mathfrak{c}$; by construction of S, the set S' has positive outer measure.

THEOREM 3'. *If* $\mathfrak{c} = \aleph_1$, *then each graph* G_M *contains either an infinite complete graph or an independent set of vertices of second category.*

Theorem 3' follows from a theorem of Lusin [7, Theorem I] which states that the continuum hypothesis implies the existence of a set S of power $\mathfrak{c}$ that meets every set of first category in a set which is at most denumerable.

Let I be a σ-ideal of subsets of M, and let $M \notin I$; that is, let I be a collection of sets A_α such that every countable union of sets of I is again in I, such that every subset of a set of I is in I, and such that M is not in I. We shall say that I *has the property* P *provided it contains a transfinite sequence* $\{B_\beta\}$ $(0 < \beta < \Omega_{\mathfrak{c}})$ *of sets such that each set of* I *is contained in at least one of the sets* B_β. By means of this concept, we now obtain a proposition which contains Theorems 3 and 3' as special cases.

THEOREM 3". *If* $\mathfrak{c} = \aleph_1$ *and if the* σ*-ideal* I *has property* p, *then each graph* G_M *contains either an infinite complete subgraph or an independent set which is not in* I.

To prove this theorem, form a nondecreasing transfinite sequence $\{A_\alpha\}$ $(0 < \alpha < \Omega_{\mathfrak{c}})$ in I such that each set in I is contained in at least one of the A_α. Let $\{x_\alpha\}$ be a transfinite sequence of distinct points such that $x_\alpha \notin A_\alpha$, and let G denote the subgraph of G_M which is determined by the set $\{x_\alpha\}$. If G contains no infinite complete subgraph, it contains an independent set S' of power $\mathfrak{c}$; clearly, none of the sets of I contains S'.

(*Added March 9, 1960:* Theorem 3 is a special case of Theorem 4 of [5]; but the proof of the latter theorem is more complicated.)

For an arbitrary σ-ideal, Theorem 3" need not hold. Indeed, Erdös and Rado [5] have constructed a graph G whose set of vertices has power $\mathfrak{c}$, which has no triangle, and which has chromatic number $\mathfrak{c}$. The independent sets of G generate a σ-ideal for which the conclusion of Theorem 3" is false.

Consider now a partition $M = A \cup B$, where A has measure 0 and B is of first category. Let the edge (x, y) belong to G_M provided $x \in A$ and $y \in B$. Then G_M contains no triangle and no independent set which is both of second category and of positive outer measure. This example should be considered in the light of Theorem 6 of [3, VI, p. 253].

THEOREM 4. *Suppose that a graph* G_M *has the following property: for some finite* n, *there do not exist sets* $\{x_i\}$ $(1 \leq i \leq n)$ *and* $\{y_j\}$ $(1 \leq j < \omega)$ *of vertices such that all the edges* (x_i, y_j) *are in* G_M. *Then* G_M *has a set of independent vertices which is of second category and of positive outer measure.*

Let $\{S_\alpha\}$ $(\alpha < \Omega_c)$ be the family of all sets of type G_δ and measure 0 and of all sets of type F_σ and first category. To prove Theorem 4, we shall construct, by transfinite induction, an independent set which is not contained in any of the sets S_α.

Suppose that we have already succeeded in constructing an independent set $\{z^\gamma\}$ $(\gamma < \beta)$ with $z^\gamma \notin S_\gamma$. If there exists a $z^\beta \notin S_\beta$ which is not connected with any z^γ $(\gamma < \beta)$, our construction proceeds. If on the other hand there exists no such z^β, our construction is stopped; in this case we delete from M the set $\{z^\gamma\}$ $(\gamma < \beta)$, and we begin the construction anew.

If the construction is stopped only finitely often, we obtain the required independen set and thus prove our result. Otherwise, we begin 2n - 1 times, and there are at least n occasions on which the construction stops because of one of the sets of type G_δ (or F_σ). We choose n such sets of the same type, denote them by S_{β_i} $(1 \leq i \leq n)$, and write $\{x_i^\gamma\}$ $(0 < \gamma < \beta_i)$ for the set of points that is deleted at the time of the stoppage occasioned by S_{β_i}.

Let C denote the complement of the union of the n sets S_{β_i}. Each point y of C is connected with one point of each of the n sets $\{x_i^\gamma\}$ $(0 < \gamma < \beta_i)$; in other words, it is connected to each point of an n-tuplet $\{x_i^{\gamma_i}\}$ $(1 \leq i \leq n$; here γ_i depends on y). Since each of the n ordinals β_i is less than Ω_c, fewer than $\mathfrak{c}$ different n-tuplets are involved; and since the n sets S_{β_i} are either all of first category or all of measure 0, the set C has cardinality $\mathfrak{c}$. Therefore, there exists a sequence $\{y_j\}$ $(0 < j < \omega)$ of points each of which is connected to each element of some n-tuplet $\{x_i^{\gamma_i}\}$ $(0 < i \leq n$; γ_i independent of j). The existence of such a sequence $\{y_j\}$ contradicts the hypothesis of Theorem 4, and our proof is complete.

Our proof makes no reference to any of the properties of the cardinal number $\mathfrak{c}$. If we assume that $\mathfrak{c}$ is regular, the proof gives the following result: Each G_M either contains, for each $n < \omega$, a subgraph $\{x_i\} \cup \{y_\alpha\}$ $(1 \leq i \leq n$; $\alpha < \Omega_c)$ such that each pair (x_i, y_j) is connected; or it has an independent set of vertices which is of second category and of positive outer measure.

We are unable to decide whether it is true that each G_M contains either a subgraph $\{x_i\} \cup \{y_j\}$ $(1 \leq i < \omega,\ 1 \leq j < \omega)$ such that each pair (x_i, y_j) is connected, or else an independent set of vertices which is of second category and of positive outer measure.

The method used in the proof of Theorem 4 yields also a stronger result:

THEOREM 5. *For any* $\mathfrak{m} < \mathfrak{c}$, *let* $\{I_\alpha\}$ $(0 < \alpha < \Omega_\mathfrak{c})$ *be a collection of* σ*-ideals with property* P. *Then each graph* G_M *either contains, for every* $n < \omega$, *a subgraph* $\{x_i\} \cup \{y_\alpha\}$ $(1 < i \leq n,\ 1 < \alpha < \Omega_\mathfrak{m})$ *such that each pair* (x_i, y_α) *is connected, or it has an independent set of vertices which is not contained in any of the* σ*-ideals* I_α.

Without property P, we are unable to prove this, even with $n = m = 2$. In fact, the result may very well not hold, since it seems likely that there exists a graph G_M which does not contain a quadrilateral and whose chromatic number is uncountable; the independent sets of such a graph would constitute a counterexample to the proposed extension of Theorem 5.

It is not clear whether Theorem 5 remains true for $\mathfrak{m} = \mathfrak{c}$; the proof certainly breaks down.

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