A REMARK ON SOME ALMOST PERIODIC COMPACTIFICATIONS

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1. The author is indebted to K. de Leeuw for raising the following, at first glance rather bizarre, question: if G is a noncompact, locally compact abelian group, with almost periodic compactification G^*, *is* G^* *the Stone-Čech compactification* $\beta(G^* \setminus G)$ *of* $G^* \setminus G$? (As usual, we view G as a subset of G^*.) At least when the character group $G^\wedge$ of G is not totally disconnected, the answer is *affirmative* (when $G^\wedge$ is totally disconnected, our approach simply fails).

Actually de Leeuw's question is not at all unnatural, since G forms a rather small part, and thus $G^* \setminus G$ a rather large part, of the "large" space G^*, as is more or less well known. For example, Borel subsets of G, that is, elements of the σ-ring generated by compact sets, are of G^*-Haar measure zero, so that, if G is σ-compact, G itself is of G^*-Haar measure zero; a special proof for $G = R$ appears in [2, Thm.4.3], but one can argue that if a Borel set E of G (automatically a Borel set in G^*) is of positive G^*-Haar measure, then $E - E$ has interior in G^*, so that G is imbedded homeomorphically in G^*. As a dense locally compact subgroup, G must fill out all of G^*, and $G = G^*$ is both compact and noncompact.

Since there are few tools available for showing a compact space to be the Stone-Čech compactification of a given subspace, there is probably no need to apologize for our use of the known structure of locally compact abelian groups; and while the result may be classed as a curiosity, it seems worth recording.

The notation used below is standard, as in [3], [4]; however we shall speak of the "direct product," where Kaplansky [3] uses "complete direct sum," for topological suggestiveness (if H is a compact group and we express $H^\wedge$ as a (weak) direct sum, there is a dual representation of H as a direct product, *topologically* and algebraically). Finally, we shall let G^d represent the discretized version of G, so that $G^* = G^{\wedge d \wedge}$ [4, p. 170].

2. Let $\{X_\nu\}$ be an uncountable set of compact Hausdorff spaces, $b = \{b_\nu\}$ an element of the topological product $X = \Pi X_\nu$, and X^b the subspace of X formed by all elements $x = \{x_\nu\}$ with $x_\nu \neq b_\nu$ for at most countably many ν. Then [1, Thm. 2] $\beta(X^b) = X$; therefore clearly $X^b \subset Y \subset X$ implies $\beta(Y) = X$.

Now suppose that G is our noncompact but locally compact abelian group, and that we can represent G^* as a direct product of uncountably many compact groups. (Such a representation is not always possible if $G^\wedge$ is totally disconnected; for example, if $G^\wedge$ is the compact group of p-adic integers, an algebraically indecomposible group, $G^* = G^{\wedge d \wedge}$ is not a product.) Then it will suffice to show that

$$G^* \neq G + G^{*o} \tag{2.1}$$

where G^{*o} is the subgroup of the product G^* consisting of all elements with at most countably many coordinates different from 0. For then $G^* \setminus G$ contains a coset of G^{*o}, and the result of [1] cited above applies.

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In order to obtain a direct product representation for G^*, we appeal to the known structure of $G^\wedge$ [5, p. 110]: $G^\wedge = R^p \times H$, where H contains a compact-open subgroup. If $G^\wedge$ is not totally disconnected, then either (1) $p > 0$ or (2) G contains a nontrivial compact connected subgroup. We consider these cases separately.

Consider case (1), where $G^\wedge = R \times F$. Since R is algebraically an uncountable direct sum ΣQ_ν ($Q_\nu = Q$, the group of additive rationals), we can write

$$G^{\wedge d} = (\Sigma Q_\nu^d) \times F^d.$$

And since G^* is simply the dual of $G^{\wedge d}$, we have $G^* = (\Pi Q_\nu^{d\wedge}) \times F^{d\wedge}$. Now let G^{*o} have the meaning indicated above. If (2.1) fails, then for each element χ of $G^* = G + G^{*o}$, there exists a countable set J of indices ν for which χ is continuous on the subspace $(\Sigma_{\nu \notin J} Q_\nu) \times \{0\}$ of $G^\wedge = R \times F$, since χ coincides with some $g \in G$ on such a subspace. But we can define a character χ of $G^{\wedge d}$ for which this fails; simply choose a character α of Q^d which is not continuous in the euclidean topology, and set $\chi(\{q_\nu\}, f) = \Pi \alpha(q_\nu)$ (a finite product).

Now consider case (2), where $G^\wedge$ contains some nontrivial compact connected group H. Here $H^\wedge$ is torsion-free, and has a countable torsion-free factor group $H^\wedge/K$. Indeed, choose a maximal independent subset $\{x_\alpha\}$ of $H^\wedge$, and for a fixed α, let

$$K = \left\{ x\colon x \in H^\wedge,\ nx = \sum_{i=1}^{m} n_i x_{\alpha_i} \text{ for some integers } n, n_i, \text{ with } \alpha_i \neq \alpha \text{ when } i = 1, 2, \cdots, m \right\}.$$

Clearly K is a subgroup for which $nx \in K$ implies $x \in K$, so that $H^\wedge/K$ is torsion-free and infinite (since it contains the infinite cyclic group generated by the coset $\bar{x}_\alpha$). But $H^\wedge/K$ is countable as well since, by maximality, for $x \in H^\wedge$, $nx - mx_\alpha \in K$ for some n, m (or $n\bar{x} = m\bar{x}_\alpha$), and this determines uniquely the coset $\bar{x}$ (since $H^\wedge/K$ has no torsion).

Therefore H contains a nontrivial compact connected and *metric* group $K^\perp = (H^\wedge/K)^\wedge$, and we may as well assume that H is metric. Since the compact connected group H is divisible, it is algebraically a direct summand of G, and moreover it is itself a direct sum ΣH_ν, where each H_ν is isomorphic to Q or $Z(p^\infty)$ for some prime p, and the indices are uncountable [3, pp. 55, 8, and 10]. Consequently we can write $G^{\wedge d} = (\Sigma H_\nu^d) \times F^d$, and $G^* = (\Pi H_\nu^{d\wedge}) \times F^{d\wedge}$. Again, if (2.1) fails, for each χ in G^* there is a countable J for which χ is continuous on the subspace of $G^\wedge$ formed by $(\Sigma_{\nu \notin J} H_\nu) \times \{0\}$. We can thus obtain our contradiction by showing that for each ν there is a χ_ν in $H_\nu^{d\wedge}$ which is not continuous in the relative topology on H_ν, and by setting $\chi(\{h_\nu\}, f) = \Pi \chi_\nu(h_\nu)$. To obtain the χ_ν, we need the following

LEMMA. *Let H_0 be an infinite algebraic subgroup of a metric compact abelian group. Then there is a χ in $H_0^{d\wedge}$ which is not continuous in the relative topology.*

If not, each χ in $H_0^{d\wedge}$ is uniformly continuous relative to the group structure, and thus it extends to a character of H_0^-. Consequently, we can identify H_0^- and H_0^{d*}; since H_0^- is compact metric, it has a countable dual; on the other hand $H_0^{d\wedge}$ is an infinite compact group, therefore uncountable (since, by category, locally compact countable groups are discrete), so that $H_0^{d\wedge d} = H_0^{d*\wedge} = H_0^{-\wedge}$ is also uncountable. This is the desired contradiction.

Finally we note that (2.1) implies $G^* \setminus G$ is pseudo-compact [1], and thus that no element of $C(G^*)$ assumes its maximum modulus only within G.

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