THE DISTRIBUTION OF PRIME ENDS

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1. INTRODUCTION

Carathéodory defined and classified the prime ends of a simply connected domain, and he raised the question how the four kinds of prime ends of a domain can be distributed (see [1], especially the last paragraph of the Introduction). Weniaminoff was the first to show that the space of prime ends of a simply connected domain can not contain an arc of prime ends of the second kind [10]. The question was further pursued by Collingwood, who proved that the prime ends of the second and fourth kinds (that is, the prime ends with subsidiary points) constitute a set of first category [2, p. 349]. For the special case where only prime ends of the first and second kinds are present, Lohwater and I showed that the prime ends of the second kind form a set of type F_σ (and of first category), and that this result is "best possible" [6, p. 8]. Recently, I solved the problem for the special case where only prime ends of the first three kinds are present [7, p. 50]. Now I shall give the solution for the general case.

2. DEFINITIONS

It will be convenient to have at hand both Carathéodory's classification of prime ends and Lindelöf's function-theoretic interpretation of the classification. Let P be a prime end of a simply connected domain B, and I(P) its impression, that is, the set of boundary points of B which is naturally associated with the equivalence class constituting the prime end P. A point p in I(P) is a *principal point* of P provided the prime end has a chain of crosscuts converging to p (in the Euclidean metric); it is a *subsidiary point* of P if it is not a principal point of P. A prime end is of the *first, second, third* or *fourth kind* according as its impression consists of

(i) only one point (necessarily a principal point),

(ii) one principal point and some subsidiary points,

(iii) more than one principal point and no subsidiary points,

(iv) more than one principal point and some subsidiary points.

For a more complete discussion and an illustration, see [7, p. 47].

If the function $\phi(\zeta)$ maps the unit disk $|\zeta| < 1$ conformally onto a simply connected domain B, it induces a one-to-one correspondence between the set of points $\zeta = e^{i\theta}$ $(0 \leq \theta < 2\pi)$ and the set of prime ends of B [1, p. 350]. Also, the radial cluster set of ϕ at $e^{i\theta}$ coincides with the set of principal points of the corresponding prime end [5, p. 28]. Therefore the prime end corresponding to $e^{i\theta}$ is of the first or second kind if and only if the radial limit of ϕ at $e^{i\theta}$ exists, and it is of the first or third kind if and only if the radial cluster set of ϕ at $e^{i\theta}$ is identical with the complete cluster set of $e^{i\theta}$.

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Corresponding to each fixed, simply connected domain B in the z-plane, we denote by U(B) (or simply by U) the space of prime ends of B. In U, we assume the natural topology of Urysohn [9, p. 235], in other words, the topology which the ordinary Hausdorff topology on the unit circle induces in U(B) under a conformal mapping of the unit disk onto B. By U_q (q = 1, 2, 3, 4) we denote the sets of prime ends of the four kinds, respectively.

3. THE DISTRIBUTION THEOREM

THEOREM 1. *Let* E_1, E_2, E_3, E_4 *be four disjoint point sets whose union is the unit circle* C. *In order that there exists a simply connected domain* B *such that some homeomorphism between* C *and the space of prime ends of* B *maps* E_1, E_2, E_3, E_4 *onto* U_1, U_2, U_3, U_4, *respectively, it is necessary and sufficient that the following three conditions are satisfied.*

(3.1) E_1 *is a set of type* G_δ;

(3.2) $E_1 \cup E_2$ *is a set of type* $F_{\sigma\delta}$, *and on each arc of* C *it has the power of the continuum;*

(3.3) $E_1 \cup E_3$ *is a residual set of type* G_δ.

(To prevent any possible misunderstanding, I point out that Theorem 1 solves only the *topological* question that Carathéodory had in mind, in other words, Problem 2 on p. 49 of [7]. The much more difficult geometrical question (Problem 1 on p. 48 of [7]) remains open: To find necessary and sufficient conditions on the partition of C into four disjoint sets such that these sets correspond to U_1, U_2, U_3, U_4, respectively, under the homeomorphism induced by some *conformal* mapping between the unit disk and some appropriate simply connected domain. The geometrical problem has been solved only for the case where both E_3 and E_4 are empty [6, p. 7].)

To prove the necessity of the three conditions, we shall simply show that the conditions are satisfied by the sets E_q on C which correspond to the sets U_q (q = 1, 2, 3, 4) in U(B), under a conformal mapping of a simply connected domain B onto the unit disk. Since the conditions are entirely topological, this will establish their necessity for the existence of any homeomorphism between C and U(B). The details of our proof will be straightforward (indeed, they could have been described forty years ago), and we shall give them in the present section. The proof of the sufficiency requires a rather elaborate synthesis, and it will occupy the remainder of the paper.

The necessity of conditions (3.1) and (3.2) is already established [7, Theorems 1 to 3]. Collingwood [2, Theorem 4] showed that the set $E_2 \cup E_4$ is of first category, in other words, that $E_1 \cup E_3$ is a residual set. We shall now show that the (complementary) sets $E_1 \cup E_3$ and $E_2 \cup E_4$ are of types G_δ and F_σ, respectively.

Consider a fixed domain B, an arbitrary point z_0, and a positive number d. Let $R(z_0, d)$ denote the set of prime ends of B that have a principal point in the closed disk $|z - z_0| \leq d$. We begin by showing that $R(z_0, d)$ is a set of type G_δ in U(B).

Let the function ϕ represent a conformal mapping of the unit disk D upon B, let *m* denote a natural number, and let $re^{i\theta}$ be a point in D such that

$$|\phi(re^{i\theta}) - z_0| < d + 1/m.$$

Then, if A is a sufficiently small arc on the circle $|\zeta| = r$, with its midpoint at $re^{i\theta}$, the inequality $|\phi(\zeta) - z_0| < d + 1/m$ holds for all ζ on A. It follows that if $0 < \rho < 1$, then the values θ for which there exists an r $(\rho < r < 1)$ such that

$$|\phi(re^{i\theta}) - z_0| < d + 1/m$$

constitute an open set $E(\rho, m)$. As $\rho \to 1$, the set $E(\rho, m)$ decreases monotonically toward a set E(m) of type G_δ. This set contains all values θ for which

$$\liminf_{r\to 1} |\phi(re^{i\theta}) - z_0| < d + 1/m\,,$$

and it contains no values θ for which

$$\liminf_{r\to 1} |\phi(re^{i\theta}) - z_0| > d + 1/m\,.$$

Clearly, a value θ lies in the set $\bigcap_{m=1}^{\infty} E(m)$ if and only if

$$\liminf_{r\to 1} |\phi(re^{i\theta}) - z_0| \leq d\,,$$

and therefore $R(z_0, d)$ is of type G_δ.

Next, let $T(z_0, n)$ denote the set of prime ends whose impression meets the closed disk $|z - z_0| \leq 1/n$. By an elementary argument, $T(z_0, n)$ is closed. Now the set $S(z_0, n)$ of prime ends whose impression meets the closed disk $|z - z_0| \leq 1/n$ and whose set of principal points does not meet the larger closed disk $|z - z_0| \leq 2/n$ is the intersection of $T(z_0, n)$ with the complement (relative to U) of $R(z_0, 2/n)$, and it is therefore of type F_σ.

Finally, let $\{z_k\}$ denote a denumerable point set, dense in the plane. Since

$$U_2 \cup U_4 = \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} S(z_k, n)\,,$$

the proof of the necessity of condition (3.3) is complete.

4. SUFFICIENCY (SKETCH OF PROOF)

We now begin a construction which shows that to each decomposition of the unit circle C into four disjoint sets E_q satisfying the three conditions in Theorem 1 there corresponds a simply connected domain B such that $U_q \sim E_q$ (q = 1, 2, 3, 4), under some homeomorphism between the space U(B) and the circle C.

To simplify the typography, we write E_{qr} and E_{qrs} for

$$E_q \cup E_r \qquad \text{and} \qquad E_q \cup E_r \cup E_s\,,$$

respectively. When the letter E is involved, subscripts beyond a "decimal point" will be used to denote elements of a sequence of sets; for example, $E_{34.ij}$ will denote the element of rank (i, j) in a certain double sequence of sets associated with E_{34}.

By condition (3.1), the set E_{234} is of type F_σ, and therefore, by a decomposition theorem of Sierpiński [8, pp. 321-323], it is the union of disjoint sets $E_{234.i}$ $(i = 1, 2, \cdots)$, where $E_{234.1}$ is the interior of E_{234}, and where each of the sets $E_{234.i}$ $(i = 2, 3, \cdots)$ is closed (and necessarily nowhere dense). We deal first with the set $E_{234.2}$, then with $E_{234.3}$, and so forth; the treatment of $E_{234.1}$ will come last.

With respect to $E_{234.2}$, we subject the boundary of the unit disk D to inward deformations, by replacing certain open arcs in the complement of $E_{234.2}$ by appropriate continuous open arcs in D. (In this context, we state here once and for all that a "continuous open arc" is the image of the interval (0, 1) under a continuous mapping; no claim of uniform continuity is implied.) The substitutions transform D into a domain B which is contained in D and has a prime end of the required kind, corresponding to each point of $E_{234.2}$. We map each of the replaced boundary arcs upon its substitute, and then we continue the process by giving similar treatment (on a progressively smaller scale) to each of the sets $E_{234.i}$ $(i = 3, 4, \cdots)$. The space of prime ends of the resulting domain has the appropriate structure, except for certain open arcs of prime ends (all of the first kind) that correspond to the components of $E_{234.1}$. To complete the proof, we subject each of the exceptional boundary arcs to alternate inward and outward deformations, according to a plan described in Section 8.

5. THE HOOKS FOR THE SET $E_{34.2}$

If the set E_{234} is open, the construction described in Sections 5 to 7 can be omitted; if the set is not open, we may assume that $E_{234.2}$ is not empty. We write $E_{34.2} = E_{34} \cap E_{234.2}$ (with similar meanings attached to $E_{2.2}$, $E_{3.2}$, $E_{4.2}$), and if $E_{34.2}$ is not empty, we carry out the following construction (if $E_{34.2}$ is empty, we go directly to the phase described in Section 6).

By condition (3.2), $E_{34.2}$ is of type $G_{\delta\sigma}$; for it is the complement, relative to the closed set $E_{234.2}$, of the set $E_{34} \cap E_{234.2}$, which is of type $F_{\sigma\delta}$. In other words, $E_{34.2}$ is the union of sets $E_{34.2j}$ $(j = 2, 3, \cdots$; convenience dictates that here the index j should not take the value 1), each of type G_δ. Each of the sets $E_{34.2j}$ is the intersection of a decreasing sequence of open sets $E_{34.2jk}$ $(k = 1, 2, \cdots)$. By (3.3), $E_{3.2}$ is of type G_δ, since it is the complement of $E_{24.2}$ (of type F_σ) with respect to $E_{234.2}$. We represent it as the intersection of a decreasing sequence of open sets $E_{34.21k}$ $(k = 1, 2, \cdots$; the index 4 before the decimal point merely contributes to notational uniformity).

In some of our future operations, difficulties could arise if the complements of some of the sets $E_{34.2jk}$ should be too thin, for example, if the complement of one of the sets were to have isolated points. We therefore replace each component of each of the sets $E_{34.2jk}$ by an open subset, in the following manner. From the interior of the component, we delete a closed arc that does not meet the set $E_{234.2}$ (since $E_{234.2}$ is nowhere dense, this is possible). Between the deleted interval and each end of the component of $E_{34.2jk}$, we delete a further closed interval that does not meet $E_{234.2}$. We continue this process in such a way that the component of $E_{34.2jk}$ is replaced by an open set whose components, ordered from left to right, have the order-type of the integers, and in such a way that no two of the components have a common endpoint. When the process of partial deletion has been applied to each component of $E_{34.2jk}$, we denote by H_{2jk} the open subset of $E_{34.2jk}$ that remains. Clearly, appropriate diligence in the process guarantees that the following three conditions are satisfied.

(5.1) $$E_{34.2jk} \supset H_{2jk} \supset (E_{34.2jk} \cap E_{34});$$

(5.2) each component of H_{2jk} has length at most $1/jk$;

(5.3) each component of H_{2jk} meets the set $E_{34.2}$.

We order the components of all sets H_{2jk} (j, k = 1, 2, ⋯) into a sequence of arcs A_s (s = 1, 2, ⋯); and we treat these arcs in the order determined by the index s. First we select two closed arcs b_1 and b_1^* on the unit circle; they must be interior to the complement of $E_{234.2} \cup A_1$, and must lie to the left and right, respectively (as seen from the origin), of A_1, and within a distance 1/2 from A_1; moreover, if A_1 is a component of H_{2jk}, then no point of $H_{2jk} - A_1$ must lie between b_1 and b_2. We denote by β_1 and β_1^* the sectors of the annulus $1/2 < |z| < 1$ determined by the arcs b_1 and b_1^*, respectively. The arc b_1 is now replaced by a continuous simple arc B_1 which begins at the left end of b_1, passes through β_1 and into the disk $|z| < 1/2$, enters β_1^*, returns to the disk $|z| < 1/2$ and into β_1, and ends at the right endpoint of b_1. The union of B_1 and the arc $C - b_1$ is required to be a curve of bounded curvature; and all points of B_1 must lie between the radii of the left end of b_1 and the right end of b_1^*, respectively. Similarly, the arc b_1^* is replaced by an arc B_1^* (see Figure 1).

Our description of B_1 and B_1^* has not yet made clear how deeply the two arcs penetrate the domains β_1^* and β_1 respectively. We define as the *aperture* of B_1 the distance between the unit circle C and the arc $B_1 \cap \beta_1^*$; similarly, the aperture of B_1^* is the distance between C and $B_1^* \cap \beta_1$. Now, by condition (3.3), $E_{24.1} = \bigcup_{t=1}^{\infty} F_t$, where $\{F_t\}$ denotes an increasing sequence of closed sets. If a pair of hooks is associated with an arc A_s, and if this arc arises from a set H_{21k}, the hooks shall have aperture 1/4s; if the arc A_s arises from a set H_{2jk} $(j > 1)$, the hooks shall have aperture 1/2 - 1/4t, where t is the least index for which A_s meets the set F_t.

Suppose that for each of the indices r = 1, 2, ⋯, s - 1 the substitutions relevant to A_r have been made, and that these substitutions have produced finitely many pairs of hooks. It may happen that some of these pairs of hooks lie partly or entirely in the sector of the unit disk which is determined by A_s. However, the distance between the union of all hooks hitherto constructed and the intersection of the annulus

$$1/2 < |z| < 1$$

with the set of radii through points of $E_{234.2}$ is positive. Therefore we can select a finite number of subarcs of A_s such that each point of $A_s \cap E_{234.2}$ is contained in one of the subarcs, and such that no portion of any of these subarcs has been replaced in any of the previous substitutions. In other words, we have ample space for the construction of either a pair of hooks (of height slightly greater than 1/2) associated with the arc A_s, or else pairs of hooks associated with appropriate subarcs of A_s (that is, with all the subarcs of any importance). Incidentally, the arcs b_s and b_s^* (or their finitely many analogues) should be chosen so that each is strictly interior to a hitherto undisturbed arc of the unit circle and lies within a distance at most 2^{-s} from A_s.

When the construction has been carried out for s = 1, 2, ⋯, we map each of the arcs of the unit circle that have been replaced upon its replacement, by some sense-preserving homeomorphism g(p) whose character is not relevant to the discussion. Before going on with the construction, we shall examine the relation between the domain B into which the unit disk has been transformed and an arbitrary point p on the unit circle (or its image g(p) on the boundary of B).

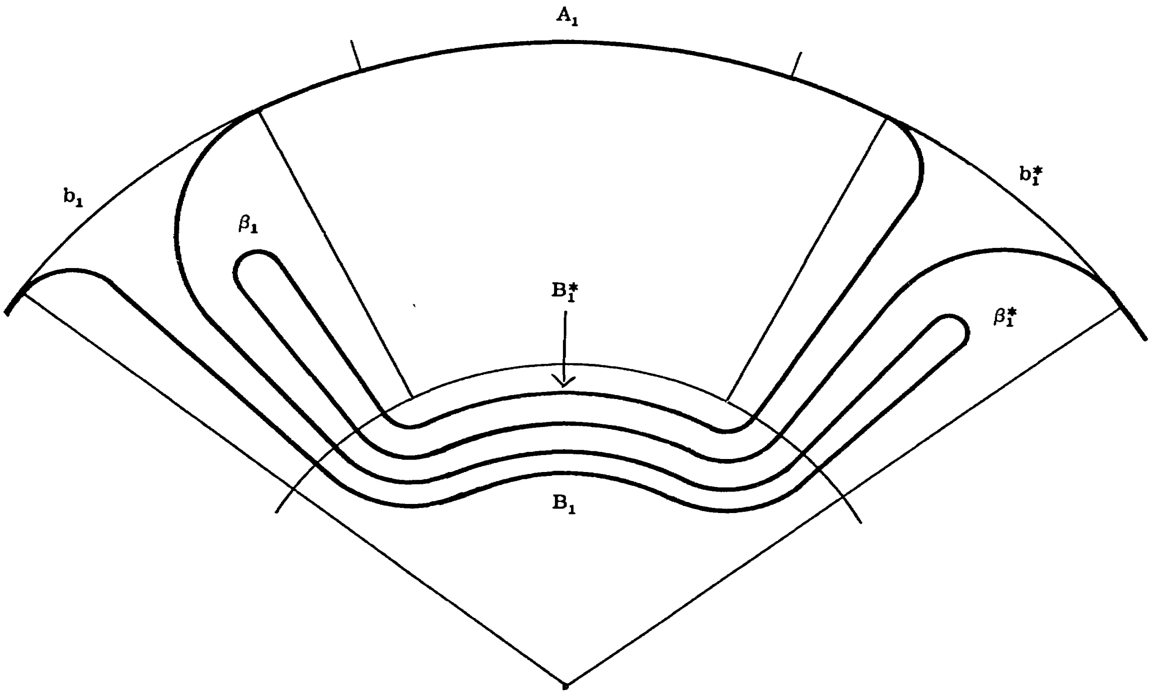

Figure 1

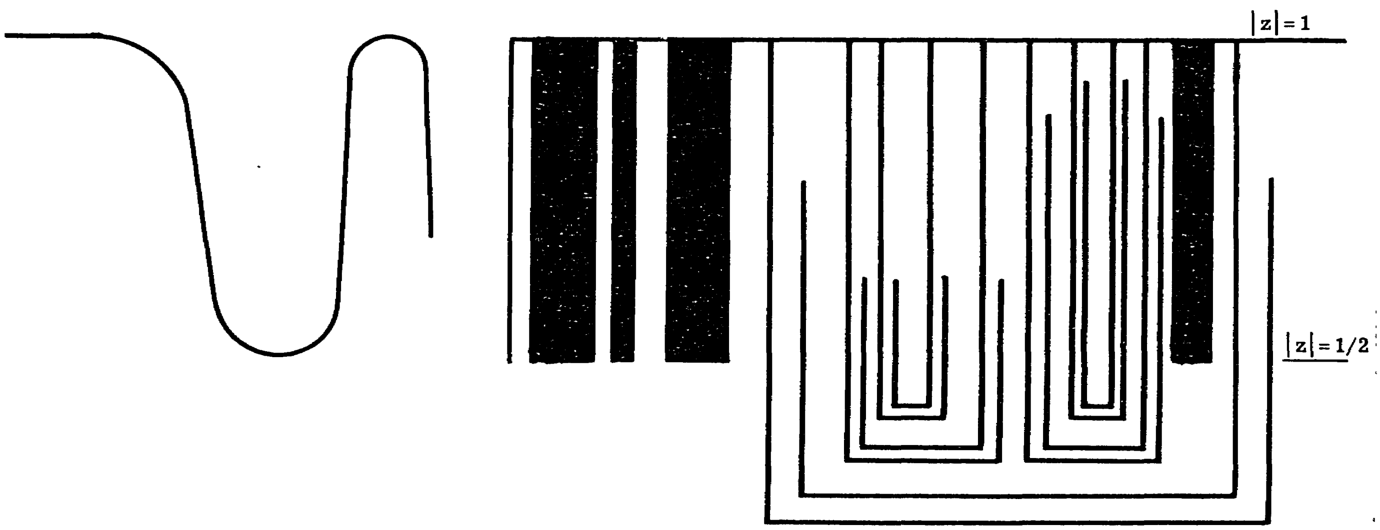

Figure 2

If $p \notin E_{234.2}$, then conditions (5.2) and (5.3) guarantee that g(p) lies either on a hook, or at one of the two ends of a hook, or on an open arc of the unit circle which has not been disturbed, so far; in each of these cases, g(p) is the impression of a prime end of the first kind.

If $p \in E_{234.2}$, then $g(p) = p$. We consider successively the three cases $p \in E_{3.2}$, $p \in E_{4.2}$, $p \in E_{2.2}$.

If $p \in E_{3.2}$, then by condition (5.1) and our construction, there exist infinitely many integers s such that p lies above a pair of hooks of aperture 1/4s. That is, p belongs to the impression of precisely one prime end; the impression of this prime end consists of a line segment of length 1/2, and all its points are principal points.

If $p \in E_{4.2}$, there exists a least integer t such that $p \in F_t$. Therefore p lies above infinitely many pairs of hooks. With at most finitely many exceptions, these pairs of hooks have aperture 1/2 - 1/4t, and therefore p belongs to the impression of a prime end of the fourth kind: a radial segment of length 1/2, whose inner portion of length 1/4t consists of principal points.

If $p \in E_{2.2}$, then at most finitely many pairs of hooks lie below p. It may happen that, from each closed arc of C that terminates at p, a pair of hooks has been suspended. In that case, the radial line segment joining the points p/2 and p is the impression of a prime end of the second kind (with p/2 as the principal point). In the other case, we need a further modification of the domain B.

6. THE WAVES FOR THE SET $E_{2.2}$

Suppose that $p = e^{i\theta_0} \in E_{2.2}$, and that p is the endpoint of a closed arc A, on the unit circle, which has not yet been modified. We consider the transformation (polar coordinates)

$$(1, \theta) \rightarrow \left(1 - \frac{1}{2}\cos^2 \frac{1}{\theta - \theta_0}, \theta\right) \qquad (\theta \neq \theta_0),$$

choose a point q of A which is carried into itself by this transformation, and apply the transformation to the arc pq of the unit circle C. When this has been done wherever necessary, we project each point p in $E_{2.2}$ to the corresponding point p/2, and each of the treated segments onto its transform. Clearly, each of the points p/2 ($p \in E_{2.2}$) is now the principal point of a prime end of the second kind, and nothing else has been disturbed, except as it has already been discussed in Section 5. Figure 2 indicates how the boundary of the domain B may appear, at this stage. In this figure, a sector of the unit disk is represented by a portion of the strip $0 < y < 1$; also, for the sake of transparency, each hook is represented by a polygonal line of three segments, and each wave (except one) by a rectangular box.

7. THE SETS $E_{234.i}$ (i = 3, 4, ···)

In the further discussion, the symbol B will always denote "the unit disk as it has been modified, so far." Except where the contrary is obvious from the context, capital letters with or without subscripts will refer either to point sets on the unit circle, or to the images of these point sets under the mappings that have carried portions of the circle onto the boundary of B.

Since the sets $E_{234.2}$ and $E_{234.3}$ are closed and disjoint, the boundary of B contains an open set S such that $S \supset E_{234.3}$ and such that the curvature of the boundary of B is uniformly bounded, on S.

If we choose the positive constant d_3 small enough (and, in any case, much smaller than the constant $d_2 = 1$), and construct segments of length d_3 on the inward normals, at all points of each component of S, these segments are mutually disjoint and lie in the interior of the modified domain. We let these segments play the role played by the radii of the unit disk, in Sections 5 and 6, and we carry out the analogous construction for $E_{234.3}$, but with the height of the hooks slightly greater than $d_3/2$, also with the restriction that the base of each hook lies within a distance at most 3^{-s} from the corresponding arc. We then map each replaced arc upon its replacement, and continue the process with $E_{234.4}$, $E_{234.5}$, $\cdots$. Clearly, if the sequence $\{d_i\}$ decreases fast enough, the domain B constructed under this program has all the desired properties, except that each point in the interior $E_{234.1}$ of E_{234} is the impression of a prime end of the first kind.

8. THE SACKS, HOOKS, WAVES, HOODS, AND SLEEVES FOR $E_{234.1}$

As matters now stand, each component of the open set $E_{234.1}$ is a continuous, simple arc of locally bounded curvature. With each point p on such an arc L we associate a point f(p) which lies on the inward normal to L at p, in such a way that the function f(p) is continuous and satisfies the inequalities

$$0 < |f(p) - p| < R(p), \tag{8.1}$$

where R(p) denotes the Euclidean distance between p and the set $\bar{L} - L$ (the bar indicates closure). We also require the quantity f(p) - p to be small enough so that all the line segments [p, f(p)] are mutually disjoint (this is possible, since the radius of curvature of L is locally bounded away from 0), and so that the set L* of points f(p) ($p \in L$) does not contain any point which, by virtue of any previous construction, is a boundary point of the domain B. The domain between the curves L and L* will be called the *sack under* L.

By condition (3.3), the set $E_{24.1}$ is the union of disjoint closed sets K_t, each nowhere dense. Corresponding to the set K_1, we now construct a family of hooks and waves, each lying inside of the sack under the corresponding component of L. The construction follows a program quite similar to the program in Sections 5 and 6, except that the hooks of small aperture (associated with the set E_3) are now omitted. All problems of "draftsmanship" are solved if we let the normal line segments [f(p), p] take over the role which in Sections 5 and 6 was played by the radii of the unit disk. The height of each hook is to be slightly greater than the maximum value of $|f(p) - p|/2$, where p runs over the arc A with which the hook is associated (no loss of generality is entailed in the assumption that the ratio between the maximum and the minimum values of $|f(p) - p|$, for any such arc A, is less than 3/2). Again, we map each replaced arc onto its replacement.

After the work for K_1 is done, we turn our attention to the open set $E_{234.1} - K_1$. From each of its components, we delete a countable set of points of E_2 whose derived set consists of the two endpoints of the component. Let M denote a component of the set that remains. The closure of M is a continuous arc of bounded curvature, and each of its points is accessible from the exterior of our domain. We now construct a *hood* over M (see Figure 3), that is, a "temporary" extension of the domain; the diameter of the hood is at least $|f(p) - p|$ and at most $2|f(p) - p|$, where p denotes the midpoint of M. A portion of the hood will be adjoined to the domain, according to a program described in the remainder of this section.

By a *near-perfect* subset of an open arc we shall mean a set which has both endpoints of the arc as limit points and which would be perfect if these two limit points were adjoined to it. By condition (3.2), the Borel set $E_{2.1}$ is locally uncountable, and a theorem of Hausdorff (see the footnote on p. 436 of [4]) implies that M has a near-perfect subset which is contained in E_2. We choose such a subset; and we denote its complement relative to M by N, and the components of N by $N_1, N_2, \cdots$. Inside of the hood over M, we draw a closed line segment T (near the far end of the hood). We replace the arc M by an arc which lies in the hood and meets the line segment T in precisely one interior subsegment (see Figure 4). We map M upon its replacement in such a way that the closure of the component N_1 of N is carried onto the point set in which the replacement meets the segment T.

On the replacement for M we choose an arc that contains N_2 and whose endpoints lie in the image of the near-perfect subset and are not endpoints of any of the arcs N_j. We replace this arc by a curve which meets the segment T in a single line segment (which does not meet the image of $\overline{N}_1$), and we map the arc onto the curve in such a way that the intersection of the curve with T becomes the image of $\overline{N}_2$. We continue the process; and we note that, after countably many steps, the original open set N has been carried onto an open subset of the line segment T in the far end of the hood. The extension which has been grafted onto the domain B has the property that each of its prime ends is of the second kind, except for the prime ends of the first kind that correspond to the open set N (and the prime ends of the first kind that correspond to the endpoints of the components of N on T; these latter prime ends will automatically be transformed into prime ends of the second kind, during our coming operations in behalf of the set K_2). We call the extension a *sleeve;* clearly, it can be constructed in such a way that it does not meet the boundary of the hood, except of course at the endpoints of M. Also, it follows from the construction that the common endpoint of any two adjacent components M and M* is now the common boundary point of two sleeves, and that it is therefore the principal point of a prime end of the second kind.

Under each of the components N_j of N (that is, under each component of the image of N on the line segment T), we construct a sack lying in B and reaching back nearly to the original arc M (Figure 5 is a simplified diagram of the distal end of a sleeve; it shows three line segments on T, together with the upper portions of the corresponding sacks). Each of the segments N_j is now treated with hooks and waves, with reference to its intersection with the subset K_2 of $E_{24.1}$. Each of the corresponding hooks and waves lies inside of the appropriate sack, and each reaches back into the "bottom" of its sack near M; this implies that if the hood over M (and consequently the sack under N_j) is a substantially distorted image of a rectangle, then the hooks and waves for K_2 are similarly distorted images of the hooks and waves in Figures 1 and 2. The expression "bottom of the sack" shall have the following meaning: Consider the domain of points lying at a distance less than some small but fixed positive number d from the arc M; clearly, we can demand that each of the sacks under the various intervals N_j meet this domain; the intersection of the domain with any sack shall be the bottom of the sack.

The components of the complement of K_2 (with respect to the open segments that lay on T, at the beginning of the last paragraph) have now been transformed into locally continuous open arcs that lie on T or in the sacks under T. From each of these open arcs we delete a countable subset of E_2 whose derived set consists of the two endpoints of the arc. Over each component M' of the open set that remains we construct a new hood; this new hood reaches out of the sleeve constructed at the first stage, and outside of the sleeve it reaches back nearly to the basal

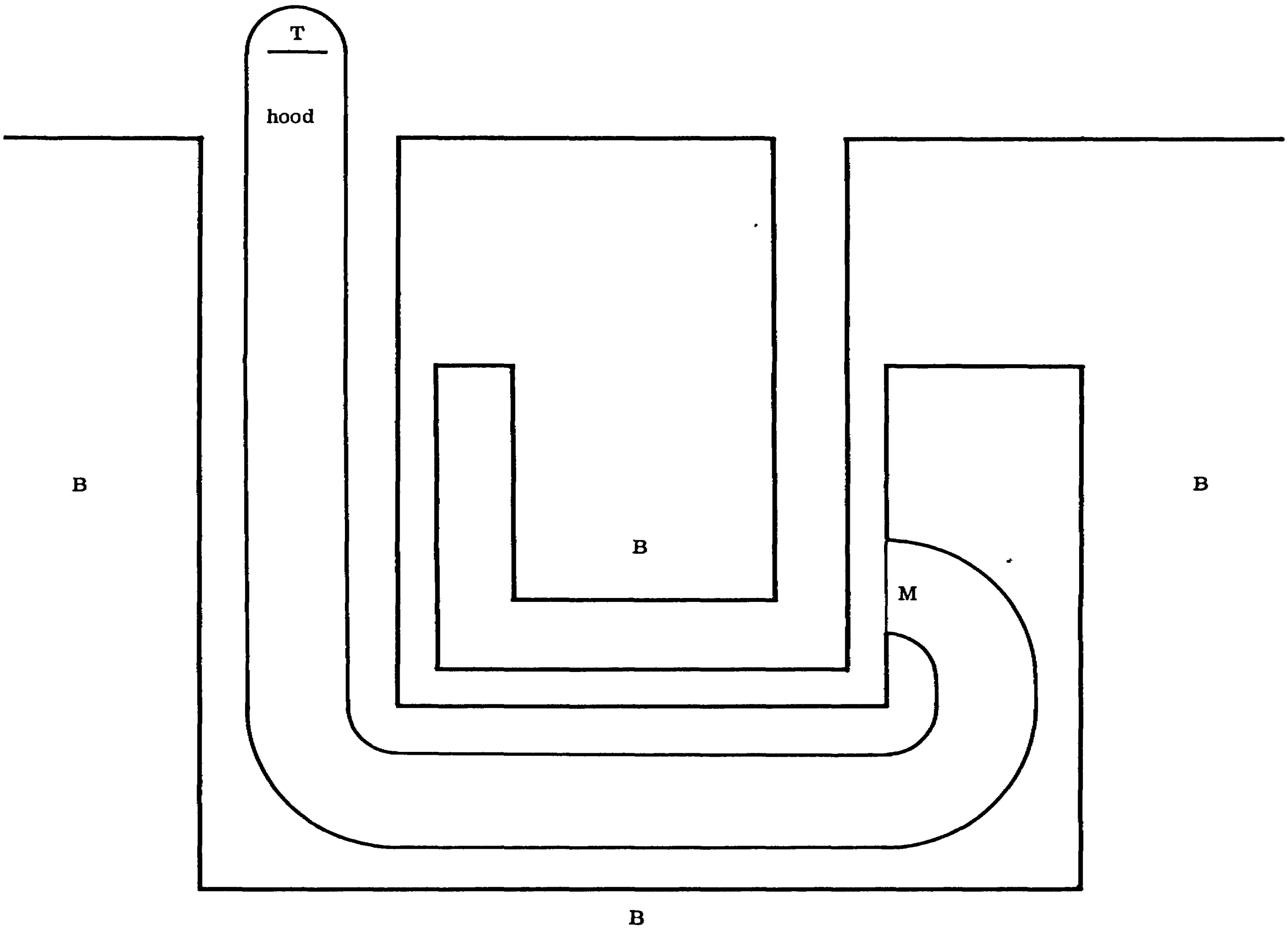

Figure 3

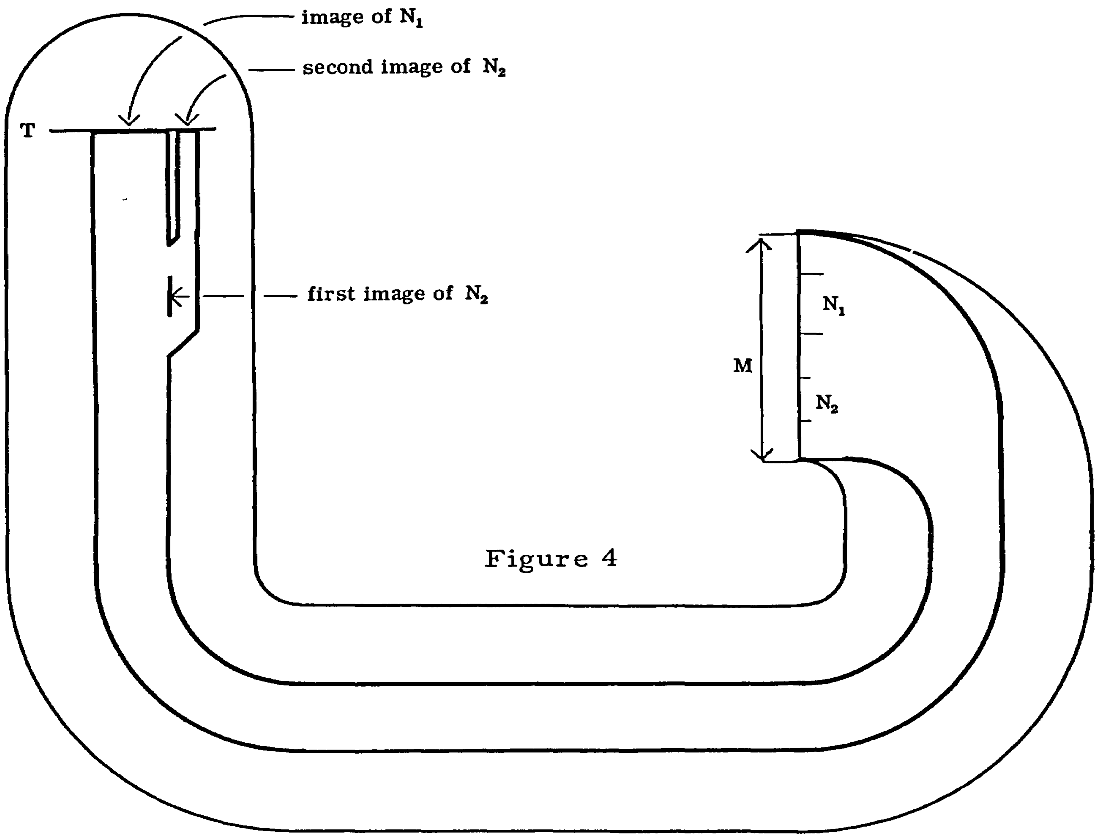

Figure 4

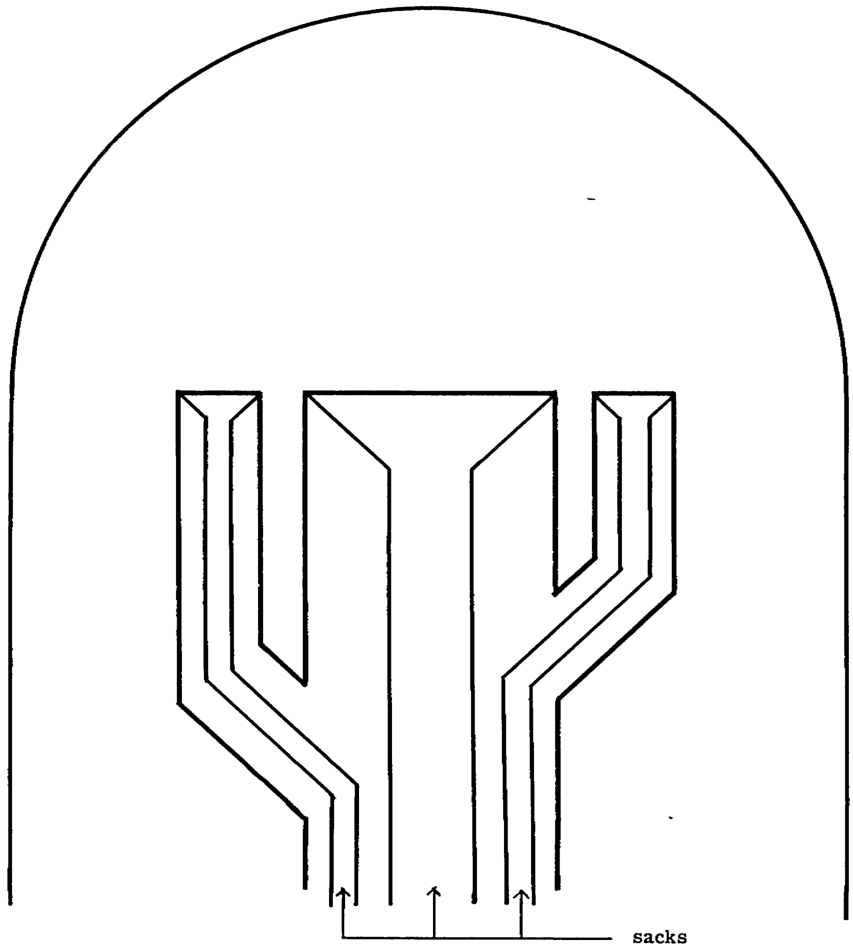

Figure 5

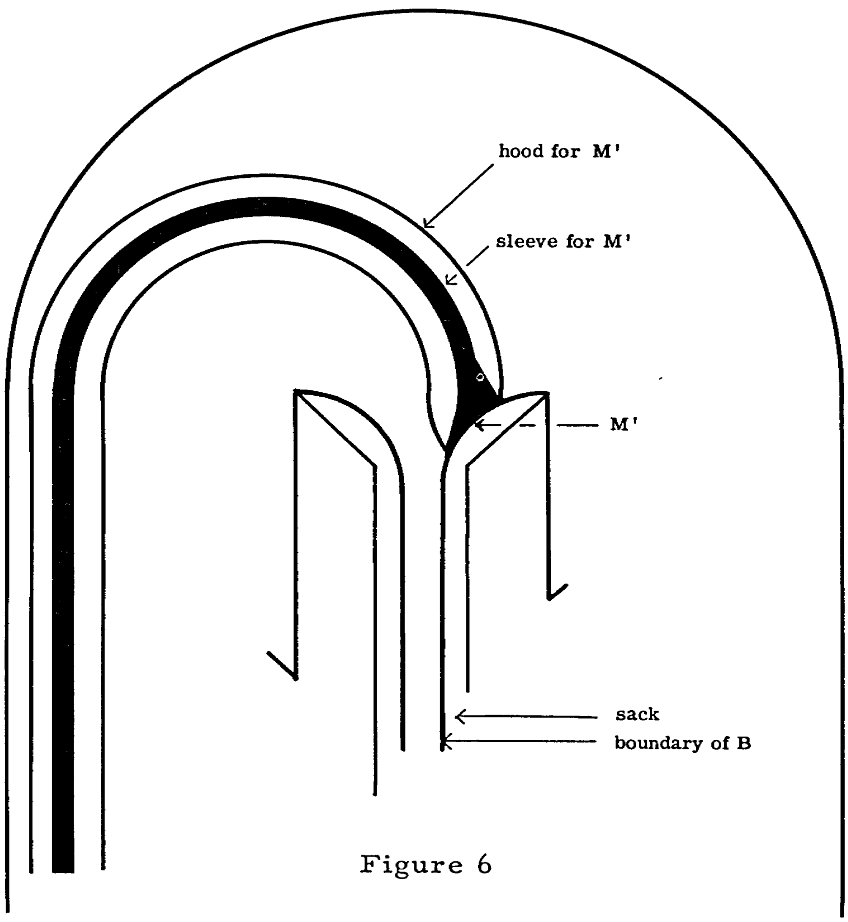

Figure 6

end of the hood in which it lies. In the distal end of the new hood (that is, near the basal end of the original hood) we construct a line segment T'. Then we delete a near-perfect subset of E_2 from M', and construct a sleeve from M' to T'. Figure 6 shows the same portion of the region as Figure 5, with the following modifications: The sleeve for M has been omitted almost entirely, and only one sack is shown, together with one component M' and the beginning of its hood and of its sleeve (solid black). We point out once more that the hood and the sleeve in Figure 6 continue nearly down to the arc M shown in Figures 3 and 4.

The process is repeated indefinitely: after the treatment of K_t, we construct hoods and sleeves for the open set that remains, and then we turn to the portion of K_{t+1} that has not been used in the deletion of any of the countable sets or near-perfect sets, in the earlier phases.

If a point p on the unit circle does not belong to $E_{234.1}$, then the inequality (8.1) guarantees that the corresponding prime end of B is not affected by the constructions relevant to $E_{234.1}$. If $p \in E_{4.1}$, then p belongs to one of the sets K_t, and the first half of the tth stage of our operations gives rise to a corresponding prime end of the fourth kind (sleeves constructed after the treatment of K_t do not affect the prime ends corresponding to K_t). If $p \in E_{2.1}$, it either belongs to one of the countable sets or one of the near-perfect sets used in our construction, or it is treated appropriately as a point in one of the sets K_t; in either case, the corresponding prime end of B is of the second kind. Finally, if $p \in E_{3.1}$, its image is carried successively to the distal end of each of infinitely many sleeves; these distal ends lie in the ends of hoods that form a nested sequence; alternate hoods in the sequence have their distal ends near the line segment T and near the arc M, respectively (see Figure 4), and therefore the corresponding prime end is of the third kind. This concludes the proof of Theorem 1.

9. FRANKL DOMAINS

We shall call a simply connected domain a *Frankl domain* provided none of its boundary points belongs to the impressions of two distinct prime ends. There exist Frankl domains without prime ends of the first kind [3].

THEOREM 2. *The construction described in Sections 4 to 8 can be carried out in such a way that* B *is a Frankl domain.*

The precautions that are necessary in the treatment of the sets $E_{234.i}$ ($i = 2, 3, \cdots$) are so simple that nothing needs to be said about them. In the constructions relevant to $E_{234.1}$, we achieve our purpose by surrounding each sack and hood by an "outer sack" or an "outer hood," and by using the outer sacks and hoods as "insulating material;" that is, by committing each outer sack once and for all to belong to the interior of B, and each outer hood to belong to the exterior of B.

REFERENCES

1. C. Carathéodory, *Über die Begrenzung einfachzusammenhängender Gebiete*, Math. Ann. 73 (1913), 323-370.

2. E. F. Collingwood, *A theorem on prime ends*, J. London Math. Soc. 31 (1956), 344-349.

3. F. Frankl, *Zur Primendentheorie*, Math. Sbornik 38 (1931) Parts 3-4, pp. 66-69.

4. F. Hausdorff, *Die Mächtigkeit der Borelschen Mengen,* Math. Ann. 77 (1916), 430-437.

5. E. Lindelöf, *Sur un principe général de l'analyse et ses applications à la représentation conforme,* Acta Soc. Sci. Fennicae, Nova Serie A, vol. 46 (1920), no. 4, 1-35 (1915).

6. A. J. Lohwater and G. Piranian, *The boundary behavior of functions analytic in a disk,* Ann. Acad. Sci. Fennicae. Series A. I., no. 239 (1957), 1-17.

7. G. Piranian, *The boundary of a simply connected domain,* Bull. Amer. Math. Soc. 64 (1958), 45-55.

8. W. Sierpiński, *Sur une classification des ensembles mesurables* (B), Fund. Math. 10 (1927), 320-327.

9. P. Urysohn, *Über ein Problem von Herrn C. Carathéodory,* Fund. Math. 6 (1924), 229-235.

10. V. Weniaminoff, *Sur un problème de la représentation conforme de M. Carathéodory,* Recueil Math. Soc. Math. Moscou 31 (1922), 91-93; Jahrb. Fortsch. Math. 48 (1921-1922), 405.

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