PERIODIC AND REVERSE PERIODIC CONTINUED FRACTIONS

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An important result in the theory of periodic simple continued fractions is a theorem of Galois [1, p. 2] (see, for example [2, v. 1, p. 76]). This theorem relates the value of a periodic simple continued fraction to the value of its reverse periodic simple continued fraction. The purpose of the present paper is to obtain a generalization of the Galois theorem which is applicable to a wider class of periodic continued fractions.

The general pure K-periodic continued fraction with nonzero partial numerators is considered in Section 1. The desired theorem is stated in terms of the conjugate points of the continued fraction which, in this case, are identical with the fixed points of the associated linear fractional transformation. In Section 2 the result of Section 1 is extended to mixed K-periodic continued fractions.

1. THE PURE PERIODIC CASE

Let

$$b_0 + \frac{a_1}{b_1} + \cdots + \frac{a_{K-1}}{b_{K-1}} + \frac{a_K}{b_0} + \frac{a_1}{b_1} + \cdots \qquad (a_j \neq 0) \tag{1.1}$$

be a pure K-periodic continued fraction with nonzero partial numerators. The associated linear fractional transformation is

$$T_K(x) = \frac{A_{K-1}\,x + a_K A_{K-2}}{B_{K-1}\,x + a_K B_{K-2}}, \tag{1.2}$$

where the fundamental recurrence formulas for A_n and B_n are

$$\begin{aligned} A_{n+1} &= b_{n+1} A_n + a_{n+1} A_{n-1}, \\ B_{n+1} &= b_{n+1} B_n + a_{n+1} B_{n-1} \qquad (n = 0, 1, 2, \cdots), \end{aligned} \tag{1.3}$$

with $A_{-1} = 1$, $B_{-1} = 0$, $A_0 = b_0$, $B_0 = 1$, and with

$$a_{n+1} = a_{j+1}, \qquad b_n = b_j \qquad (n \equiv j \pmod{K}). \tag{1.4}$$

The conjugate points for (1.1) are defined to be the fixed points, x_1 and x_2, of the associated linear fractional transformation (1.2). Since x_1 and x_2 are the solutions of $x = T_K(x)$, or

$$B_{K-1}\,x^2 + (a_K B_{K-2} - A_{K-1})x - a_K A_{K-2} = 0, \tag{1.5}$$

they are quadratic conjugates relative to this equation. The fact that $T_K(x)$ is nonsingular follows from the condition $a_j \neq 0$ and the determinant formula

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(1.6) $$A_n B_{n-1} - A_{n-1} B_n = (-1)^{n-1} a_1 a_2 \cdots a_n.$$

The reverse pure K-periodic continued fraction of (1.1) is defined to be

(1.1*)
$$b_0^* + \frac{a_1^*}{b_1^*} + \cdots + \frac{a_{K-1}^*}{b_{K-1}^*} + \frac{a_K^*}{b_0^*} + \frac{a_1^*}{b_1^*} + \cdots$$
$$= b_0 + \frac{a_K}{b_{K-1}} + \cdots + \frac{a_2}{b_1} + \frac{a_1}{b_0} + \frac{a_K}{b_{K-1}} + \cdots,$$

where, subject to (1.4),

(1.7) $$b_n^* = b_{K-n}, \qquad a_{n+1}^* = a_{K-n}.$$

The following theorem relates the conjugate points for (1.1) and (1.1*), and the values of (1.1) and (1.1*) when these continued fractions are convergent.

THEOREM 1. *For suitable ordering, the conjugate points for* (1.1) *and* (1.1*) *satisfy*

(1.8) $$x_1^* = b_0 = x_2, \qquad x_2^* = b_0 - x_1.$$

Moreover, if both (1.1) *and* (1.1*) *converge, their values are respectively* x_1 *and* x_1^*.

Proof. Let $A_{n,p}$ and $B_{n,p}$ denote respectively the n-th numerator and denominator of

$$b_p + \frac{a_{p+1}}{b_{p+1}} + \cdots + \frac{a_K}{b_0} + \frac{a_1}{b_1} + \cdots + \frac{a_p}{b_p} + \frac{a_{p+1}}{b_{p+1}} + \cdots.$$

Then [2, v. 1, p. 11], for n, p = 0, 1, 2, ⋯,

(1.9)
$$A_{n+1,p} = b_p A_{n,p+1} + a_{p+1} A_{n-1,p+2},$$
$$B_{n+1,p} = b_{p+1} B_{n,p+1} + a_{p+2} B_{n-1,p+2},$$

and

(1.10) $$A_{n,p+1} = B_{n+1,p}.$$

For the n-th numerator and denominator, A_n^* and B_n^*, of (1.1*) it is readily found that

$$A_{-1}^* = 1 = B_{0,K}, \quad A_0^* = b_0^* = b_0 = B_{1,K-1},$$
$$B_0^* = 1 = A_{-1,K}, \quad B_1^* = b_1^* = b_{K-1} = A_{0,K-1}.$$

Thus, by (1.9) and mathematical induction,

(1.11) $$A_{p-1}^* = B_{p,K-p}, \qquad B_p^* = A_{p-1,K-p} \qquad (p = 0, 1, \cdots, K).$$

In view of (1.10) this can be rewritten as

(1.12) $A^*_{p-1} = A_{p-1,K-p+1}, \quad B^*_p = B_{p,K-p-1} \quad (p = 0, 1, \cdots, K).$

The conjugate points, x_1^* and x_2^*, for (1.1*) are the solutions x^* of the equation $x^* = T_K^*(x^*)$. Formulas (1.3*), (1.11) and (1.12) can be used to show that

$$x^* = \frac{A^*_{K-1}(x^* - b_0) + A^*_K}{B^*_{K-1}(x^* - b_0) + B^*_K} = \frac{B_{K,0}(x^* - b_0) + A_{K,0}}{B_{K-1,0}(x^* - b_0) + A_{K-1,0}},$$

from which it follows by (1.3) that

$$b_0 - x^* = \frac{-a_K B_{K-2}(b_0 - x^*) + a_K A_{K-2}}{B_{K-1}(b_0 - x^*) - A_{K-1}},$$

since $A_{n,0} = A_n$ and $B_{n,0} = B_n$. Thus $b_0 - x^*$ is a fixed point for the inverse of $T_K(x)$, and the proof of the statement (1.8) is complete in case all conjugate points are finite.

It is easily seen that (1.2) has an infinite fixed point if and only if $B_{K-1} = 0$, and that (1.2) has two or one infinite fixed points according as $a_K B_{K-2} - A_{K-1}$ is or is not zero. Since $B^*_{K-1} = B_{K-1} = 0$ by (1.12), it follows from the fundamental recurrence formulas and (1.12) that

$$a_K^* B^*_{K-2} - A^*_{K-1} = -(a_K B_{K-2} - A_{K-1}),$$

and hence that (1.2) and (1.2*) have the same number of infinite fixed points. In case $x_1^* = \infty$ and $x_2^* = a_K^* A^*_{K-2}/(a_K^* B^*_{K-2} - A^*_{K-1})$ (finite), it follows as before that $b_0 - x_1^* = x_2 = \infty$ and

$$b_0 - x_2^* = a_K A_{K-2}/(a_K B_{K-2} - A_{K-1}) = x_1 .$$

Thus (1.8) holds formally in the case of infinite conjugate points.

By a well-known theorem on convergence of pure K-periodic continued fractions [2, v. 2, p. 86], the condition $B_{K-1} \neq 0$ is necessary for convergence of (1.1); moreover, if $x_2 = x_1$, (1.1) converges to x_1. In this case, (1.8) shows that $x_2^* = x_1^*$ and (1.1*) *must* converge to $x_1^* = b_0 - x_1$. In the remaining case, if (1.1) converges, its value x_1 must satisfy

(1.13) $$|B_{K-1} x_1 + a_K B_{K-2}| > |B_{K-1} x_2 + a_K B_{K-2}|.$$

From (1.5*) and (1.7),

$$x_1^* + x_2^* = (A^*_{K-1} - a_1 B^*_{K-2})/B^*_{K-1},$$

and this, together with (1.3), (1.7), (1.8), (1.11) and (1.12), yields

$$B^*_{K-1} x_i^* + a_K^* B^*_{K-2} = B_{K-1} x_i + a_K B_{K-2} \qquad (i = 1, 2).$$

When both (1.1) and (1.1*) converge, it follows from (1.13) that their values are x_1 and x_1^*, respectively, and the proof of the theorem is complete.

It might be expected that the convergence of (1.1) implies the convergence of (1.1*), but this is not the case. The reason is that, although (1.1) converges to x_1, it is possible for (1.1*) to exhibit Thiele oscillation (see [2, v. 2, p. 87]) by having the condition $A_n^* - x_2^* B_n^* \neq 0$ fail for one or more indices $n = 0, 1, \cdots, K - 2$. For example, the periodic continued fraction (1.1), where $K = 2$, $b_0 = 0$, $b_1 = 1$, and $0 < |a_1| < |a_2|$, converges to 0 while the corresponding reverse periodic continued fraction diverges ($A_0^* - x_2^* B_0^* = 0$).

COROLLARY 1 (Galois). *If a quadratic irrational* x_1 *has the simple continued fraction expansion*

$$x_1 = b_0 + \frac{1}{b_1} + \cdots + \frac{1}{b_{K-1}} + \frac{1}{b_0} + \cdots,$$

then

$$-\frac{1}{\overline{x}_1} = b_{K-1} + \frac{1}{b_{K-2}} + \cdots + \frac{1}{b_0} + \frac{1}{b_{K-1}} + \cdots,$$

where $\overline{x}_1$ *is the quadratic conjugate of* x_1.

Proof. Since (1.1) and (1.1*) are simple continued fractions, both converge and, by Theorem 1, $x_1^* = b_0 - x_2$, where

$$x_1^* = b_0 + \frac{1}{b_{K-1}} + \cdots + \frac{1}{b_1} + \frac{1}{b_0} + \frac{1}{b_{K-1}} + \cdots.$$

The proof may now be completed by noting that

$$\frac{1}{x_1^* - b_0} = -\frac{1}{x_2} = -\frac{1}{\overline{x}_1}.$$

COROLLARY 2. *Let* $\Omega(x)$ *have the periodic* C*-fraction expansion*

$$\Omega(x) \sim 1 + \frac{a_1 x^{\alpha_1}}{1} + \cdots + \frac{a_K x^{\alpha_K}}{1} + \frac{a_1 x^{\alpha_1}}{1} + \cdots, \tag{1.14}$$

in which $a_j \neq 0$ *and the* α_j *are positive integers. Then* $\Omega(x) = [P(x) + \sqrt{D(x)}]/2Q(x)$, *where* P, Q, D *are polynomials for which* $P(0) = Q(0) = D(0) = 1$, *and* D *is not square of a polynomial. Also*

$$1 - \overline{\Omega}(x) \sim 1 + \frac{a_K x^{\alpha_K}}{1} + \cdots + \frac{a_1 x^{\alpha_1}}{1} + \frac{a_K x^{\alpha_K}}{1} + \cdots, \tag{1.14*}$$

where $\overline{\Omega}(x) = [P(x) - \sqrt{D(x)}]/2Q(x)$ *is the quadratic conjugate of* $\Omega(x)$.

Proof. The statement concerning the form of $\Omega(x)$ follows from a result on periodic C-fractions [2, v. 2, p. 113]. In a sufficiently small neighborhood of $x = 0$, $|a_j x^{\alpha_j}| \leq 1/4$, and the uniform convergence of (1.14) and (1.14*) to $x_1 = \Omega(x)$ and to x_1^* is assured near $x = 0$. Application of Theorem 1 completes the proof, since $x_1^* = 1 - x_2 = 1 - \overline{\Omega}(x)$.

A similar argument suffices to prove the following result for J-fractions.

COROLLARY 3. *Let* $\omega(z)$ *have the periodic* J*-fraction expansion*

$$\omega(z) \sim z + b_0 - \frac{a_1^2}{z + b_1} - \cdots - \frac{a_K^2}{z + b_0} - \frac{a_1^2}{z + b_1} - \cdots,$$

where $a_j \neq 0$. *Then* $\omega(z)$ *is a quadratic irrational function whose quadratic conjugate function* $\overline{\omega}(z)$ *has the J-fraction expansion*

$$\overline{\omega}(z) \sim \frac{a_K^2}{z + b_{K-1}} - \cdots - \frac{a_1^2}{z + b_0} - \frac{a_K^2}{z + b_{K-1}} - \cdots.$$

2. THE MIXED PERIODIC CASE

Let C_p and D_p denote the p-th numerator and denominator of the mixed K-periodic continued fraction

$$(2.1) \qquad d_0 + \frac{c_1}{d_1} + \cdots + \frac{c_n}{d_n} + \frac{c_{n+1}}{d_{n+1}} + \cdots + \frac{c_{n+K}}{d_n} + \frac{c_{n+1}}{d_{n+1}} + \cdots,$$

where $n \geq 1$ is the smallest integer for which

$$(2.2) \qquad d_n + \frac{c_{n+1}}{d_{n+1}} + \cdots + \frac{c_{n+K}}{d_n} + \frac{c_{n+1}}{d_{n+1}} + \cdots$$

is pure K-periodic. Since (2.2) can be identified with (1.1), it is convenient to rewrite (2.1) in the form

$$(2.3) \qquad d_0 + \frac{c_1}{d_1} + \cdots + \frac{c_n}{b_0} + \frac{a_1}{b_1} + \cdots + \frac{a_K}{b_0} + \frac{a_1}{b_1} + \cdots.$$

The conjugate points fqr (2.3) are defined to be the images W_1 and W_2 of the conjugate points, x_1 and x_2, for (1.1) by

$$(2.4) \qquad W = \frac{C_{n-1}x + c_n C_{n-2}}{D_{n-1}x + c_n D_{n-2}}.$$

The following theorem relates the conjugate points for (2.3) to the conjugate points for the continued fraction $\overline{(2.3)}$ obtained from (2.3) by replacing

$$(2.5) \qquad d_{n-1} + \frac{c_n}{b_0} + \frac{a_1}{b_1} + \cdots + \frac{a_{K-1}}{b_{K-1}} + \frac{a_K}{b_0} + \cdots$$

with

$$(\overline{2.5}) \qquad \left(d_{n-1} - \frac{c_n b_{K-1}}{a_K}\right) - \frac{c_n a_{K-1}/a_K}{b_{K-2}} + \frac{a_{K-2}}{b_{K-3}} + \cdots + \frac{a_1}{b_0} + \frac{a_K}{b_{K-1}} + \frac{a_{K-1}}{b_{K-2}} + \cdots.$$

THEOREM 2. *For suitable ordering, the conjugate points for* $(\overline{2.3})$ *satisfy*

$$(2.6) \qquad \overline{W}_1 = W_2, \qquad \overline{W}_2 = W_1,$$

where W_1 *and* W_2 *are the conjugate points for* (2.3). *If* (2.3) *and* $(\overline{2.3})$ *both converge, their values are respectively* W_1 *and* $\overline{W}_1$.

Proof. By (2.4) and Theorem 1,

$$W_1 = \frac{C_{n-1}(b_0 - x_2^*) + c_n C_{n-2}}{D_{n-1}(b_0 - x_2^*) + c_n D_{n-2}},$$

where x_1^* and x_2^* are the conjugate points for (1.1*). Since the conjugate points for (1.1*) and

$$b_{K-2} + \frac{a_{K-2}}{b_{K-3}} + \cdots + \frac{a_1}{b_0} + \frac{a_K}{b_{K-1}} + \frac{a_{K-1}}{b_{K-2}} + \cdots \tag{2.7}$$

are related by

$$x_i^* = b_0 + \frac{a_K}{b_{K-1}} + \frac{a_{K-1}}{y_i^*} \qquad (i = 1, 2), \tag{2.8}$$

it follows that

$$W_1 = \frac{\left(C_{n-1} - \frac{c_n b_{K-1}}{a_K} C_{n-2}\right) y_2^* - \frac{c_n a_{K-1}}{a_K} C_{n-2}}{\left(D_{n-1} - \frac{c_n b_{K-1}}{a_K} D_{n-2}\right) y_2^* - \frac{c_n a_{K-1}}{a_K} D_{n-2}}.$$

The fundamental recurrence formulas for (2.3) and $(\overline{2.3})$ may now be used to write this last equation as

$$W_1 = \frac{\overline{C}_{n-1}\, y_2^* + \overline{c}_n \overline{C}_{n-2}}{\overline{D}_{n-1}\, y_2^* + \overline{c}_n \overline{D}_{n-2}} = \overline{W}_2 .$$

Similarly, $W_2 = \overline{W}_1$, and the proof of (2.6) is complete.

The proof of the statement about the values of (2.3) and $(\overline{2.3})$ when both are convergent begins with the observation that (1.1) must converge or else diverge to ∞ (that is, its reciprocal converges to 0), and that a similar remark applies to (1.1*). When both (1.1) and $(\overline{1.1^*})$ converge, it follows from $(\overline{2.4})$, (2.8) and Theorem 1 that the values of (2.3) and $(\overline{2.3})$ are respectively W_1 and $\overline{W}_1$. When (1.1) diverges to ∞, the fact that $B_{K-1}^* = B_{K-1} = 0$ plus the prior knowledge that (1.1*) converges or diverges to ∞ shows that (1.1*) must also diverge to ∞. Theorem 1 shows that all conjugate points of (1.1) and (1.1*) are infinite in this case, and again the form of the linear fractional transformations $(\overline{2.4})$ and (2.8) leads to the desired result, namely, that the values of (2.3) and $(\overline{2.3})$ are respectively W_1 and $\overline{W}_1$.

It should be noted that Theorem 2 remains true in case n is not the smallest integer for which (2.2) is pure K-periodic, since the proof of the theorem does not use the requirement that at least one of the conditions $c_n = a_K$, $d_{n-1} = b_{K-1}$ shall fail to hold.

Results analogous to Corollaries 2 and 3 may be obtained from Theorem 2. For example, if

$$\Omega(x) \sim 1 + \frac{c_1 x^{\gamma_1}}{1} + \cdots + \frac{c_n x^{\gamma_n}}{1} + \frac{a_1 x^{\alpha_1}}{1} + \cdots + \frac{a_K x^{\alpha_K}}{1} + \frac{a_1 x^{\alpha_1}}{1} + \cdots,$$

and if $\gamma_n = \alpha_K$ and $c_n \neq a_K$, then the C-fraction expansion for $\overline{\Omega}(x)$, the quadratic conjugate of $\Omega(x)$, can be obtained from the expansion for $\Omega(x)$ by replacing

$$1 + \frac{c_n x^{\gamma_n}}{1} + \frac{a_1 x^{\alpha_1}}{1} + \cdots + \frac{a_K x^{\alpha_K}}{1} + \frac{a_1 x^{\alpha_1}}{1} + \cdots$$

with

$$\left(1 - \frac{c_n}{a_K}\right) - \frac{\frac{c_n a_{K-1}}{a_K} x^{\alpha_{K-1}}}{1} + \frac{a_{K-2} x^{\alpha_{K-2}}}{1} + \cdots + \frac{a_1 x^{\alpha_1}}{1} + \frac{a_K x^{\alpha_K}}{1} + \cdots.$$

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