A GENERALIZATION OF THE LOTOTSKY METHOD OF SUMMABILITY

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1. INTRODUCTION AND NOTATION

In a recent paper [5], A. V. Lototsky introduced a new method of summability which possesses interesting and important properties. Later, in [2], R. P. Agnew gave simplified proofs of Lototsky's results as well as some further properties of the new method. In this paper we shall show that many results proved for the Lototsky (or L) method of summability are valid for a general class of transformations to which the L transformation belongs.

Corresponding to a sequence $\{d_n\}$ with $n \geq 1$ (for all sequences in this paper, the index denoting the order of the terms assumes the values 0, 1, 2, $\cdots$, except where it is stated otherwise), the symbol $d_n!$ will denote the product $d_1 d_2 \cdots d_n$, and *not*, as is sometimes the case, the function $\Gamma(d_n + 1)$. Given a sequence $\{d_n\}$ $(n \geq 1)$, we shall denote by $\{p_{nm}\}$ $(m = 0, \pm 1, \pm 2, \cdots;\ n = 0, 1, \cdots)$ the double sequence defined by

$$(1.1)\qquad \begin{cases} p_{00} = 1\,, \\ (x + d_n)! \equiv \prod_{m=1}^{n} (x + d_m) \equiv \sum_{m=0}^{n} p_{nm} x^m\,, \\ p_{nm} = 0 \quad \text{for } m > n \text{ and } m < 0\,. \end{cases}$$

It is easy to see that

$$(1.2)\qquad p_{n+1,m} - d_{n+1} p_{nm} = p_{n,m-1} \qquad \text{for } m = 0, \pm 1, \cdots;\ n = 0, 1, \cdots.$$

2. THE $[F, d_n]$ TRANSFORMATIONS

Suppose a fixed sequence $\{d_n\}$ $(n \geq 1)$ satisfying $d_n \neq -1$ for $n \geq 1$ is given. Then $\{t_n\}$, the $[F, d_n]$ transform of $\{s_n\}$, is defined by

$$t_0 = s_0\,.$$

$$t_n = [(1 + d_n)!]^{-1} \sum_{m=0}^{n} p_{nm} s_m \qquad (n \geq 1)\,.$$

The L transformation was defined as a transformation which, applied to a sequence $\{s_n\}$ defined for $n \geq 1$, yields a transform $\{t_n\}$ defined for $n \geq 1$, also. Now for

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$d_n = n - 1$, the $[F, d_n]$ transform $\{t_n\}$ with the restriction $n \geq 1$ is identical with the L transform $\{t_n\}$ of $s_1, s_2, \cdots$.

It should be remarked here that the numbers $\{p_{nm}\}$ corresponding to the $[F, n - 1]$ (or the L) transformation are the absolute values of the Stirling numbers of the first kind (see [4, p. 142]). In fact, some of the properties of the coefficients $\{p_{nm}\}$ for the L transformation proved in [2] are well known (see [4, Chapter 4]).

The $\left[F, d_n = \frac{1+q}{q} \right]$ $(q \neq 0)$ transformation is the $[E, q]$ (or Euler-Knopp) transformation of order q defined explicitly by

$$t_n = \sum_{m=0}^{n} \binom{n}{m} q^m (1 - q)^{n-m} s_m \qquad (n \geq 0).$$

Here we can also permit the value $q = 0$.

3. REGULARITY

As is well known, a linear transformation T defined by

$$t_n = \sum_{m=0}^{\infty} c_{nm} s_m \qquad (n \geq 0)$$

is regular if and only if the three conditions

$$\sum_{m=0}^{\infty} |c_{nm}| \leq K < +\infty \quad \text{for } n \geq n_0 \geq 0, \tag{3.1}$$

$$\lim_{n\to\infty} \sum_{m=0}^{\infty} c_{nm} = 1, \tag{3.2}$$

$$\lim_{n\to\infty} c_{nm} = 0 \quad \text{for each } m \geq 0, \tag{3.3}$$

are satisfied. In order to obtain conditions on $\{d_n\}$ for the regularity of the $[F, d_n]$ transformation, we need the following three lemmas.

LEMMA 3.1. *Let* $\{d_n\}$ $(n \geq 1,\ d_n \neq -1,\ d_n \neq 0)$ *be a fixed sequence such that*

$$\lim_{n\to\infty} \left| 1 + \frac{1}{d_m} \right| ! = +\infty \tag{3.4}$$

and

$$\frac{(1 + |d_n|)!}{|1 + d_n|!} \leq K < +\infty \quad \textit{for } n \geq 1; \tag{3.5}$$

then

$$\lim_{n\to\infty} [(1 + d_n)!]^{-1} p_{nm} = 0 \quad \textit{for each } m \geq 0. \tag{3.6}$$

Proof. Define $\delta_n = d_n^{-1}$. Obviously, for $n \geq 1$,

$$p_{n,0} = \frac{d_n!}{(1 + d_n)!} = [(1 + \delta_n)!]^{-1}, \tag{3.7}$$

and by (3.4),

$$\lim_{n\to\infty} p_{n,0} = 0.$$

Define $\{\sigma_n\}$ $(n \geq 1)$, the Σ transform of $\{s_n\}$ $(n \geq 1)$, by

$$\sigma_n = \frac{(1 + \delta_1)\,\delta_1\, s_1 + (1 + \delta_1)!\,\delta_2\, s_2 + (1 + \delta_2)!\,\delta_3\, s_3 + \cdots + (1 + \delta_{n-1})!\,\delta_n\, s_n}{(1 + \delta_n)!} \tag{3.8}$$

$$(n \geq 1).$$

From (3.4) and (3.5) it follows, because $\dfrac{1 + |d_m|}{|1 + d_m|} = \dfrac{1 + |\delta_m|}{|1 + \delta_m|}$, that the Σ transformation is regular. We shall complete the proof in two steps.

a) In (3.8), choose $s_1 = (1 + \sigma_1)^{-1}$ and $s_n = p_{n-1,0}$ for $n > 1$. Then, from the regularity of the Σ transformation and (3.7), we obtain

$$\sigma_n = \frac{\delta_1 + \cdots + \delta_n}{(1 + \delta_n)!} = p_{n,1} \to 0 \quad \text{as } n \to \infty. \tag{3.9}$$

b) For $n \geq k > 0$,

$$p_{nk} = \left(\frac{d_n}{1 + d_n}\right)! \sum_{1 \leq j_1 \leq j_2 \leq \cdots < j_k \leq n} \frac{1}{d_{j_1} \cdots d_{j_k}}$$

$$= [(1 + \delta_n)!]^{-1} \sum_{1 \leq j_1 < j_2 \cdots < j_k \leq n} \delta_{j_1} \cdots \delta_{j_k}.$$

If in (3.8) we choose

$$s_n = \begin{cases} 0 & \text{for } 1 \leq n < k, \\ p_{n-1,k} & \text{for } n \geq k, \end{cases}$$

then, for $n > k$,

$$\sigma_n = [(1+\delta_n)!]^{-1} \sum_{m=k}^{n} (1+\delta_{m-1})!\, \delta_m p_{m-1,k}$$

$$= [(1+\delta_n)!]^{-1} \sum_{m=k}^{n} (1+\delta_{m-1})!\, \delta_m [(1+\delta_{m-1})!]^{-1} \cdot \sum_{1 \le j_1 < \cdots < j_k < m-1} \delta_{j_1} \cdots \delta_{j_k}$$

$$= [(1+\delta_n)!]^{-1} \sum_{m=k}^{n} \delta_m \sum_{1 \le j_1 < \cdots < j_k \le m-1} \delta_{j_1} \cdots \delta_{j_k}$$

$$= [(1+\delta_n)!]^{-1} \sum_{1 \le j_1 < \cdots < j_{k+1} \le n} \delta_{j_1} \cdots \delta_{j_{k-1}} = p_{n,k+1}\,.$$

From (3.7), (3.9), the regularity of the Σ transformation, and by induction, we see that (3.6) is satisfied. *Q. E. D.*

LEMMA 3.2. *If a fixed sequence* $\{d_n\}$ ($n \ge 1$; $d_n \ne -1$ *for* $n \ge 1$; $d_n \ne 0$ *for* $n \ge n_0$) *satisfies* (3.4) *and* (3.5), *then it satisfies* (3.6).

Proof. Denote by n_1 the largest integer m such that $d_m = 0$. Then necessarily $n_1 < n_0$. If there are n_2 values of n such that $d_n = 0$ then, as a simple consideration shows,

$$p_{nm} = 0 \qquad \text{for } 0 \le m \le n_2 \text{ and } n \ge n_1\,. \tag{3.10}$$

Now for $m > n_2$ and $n > n_1$, we have

$$(x + d_n)! = x^{n_2} \prod_{m=1}^{n}{}' (x + d_m)\,, \tag{3.11}$$

where by $\Pi'^{n}_{m=1} (x + d_m)$ we denote the product of all the factors $(x + d_m)$ with $1 \le m \le n$ and such that $d_m \ne 0$. The relation (3.11) and Lemma 3.1 complete the proof.

The following is a simple consequence of the proof of Lemma 3.2.

LEMMA 3.3. *If* $\{d_n\}$ ($n \ge 1$, $d_n = -1$) *is a sequence with* $d_n = 0$ *for infinitely many* n, *then* (3.6) *is true.*

From (1.1) it follows that

$$\sum_{m=0}^{n} |p_{nm} x^m| \le (|x| + |d_n|)! \qquad \text{for } n \ge 1\,,$$

and in particular, with $x = 1$, that

$$\sum_{m=0}^{n} |p_{nm}| \le (1 + |d_n|)! \qquad \text{for } n \ge 1\,.$$

Hence

$$[\,|1 + d_n|\,!]^{-1} \sum_{m=0}^{n} |p_{nm}| \le \frac{(1 + |d_n|)!}{|1 + d_n|\,!} \qquad (n \ge 1). \tag{3.12}$$

From (3.12) it follows that if (3.5) is satisfied, then the $[F, d_n]$ transformation satisfies (3.1). We leave open the question whether the validity of (3.1) implies that of (3.5).

The following propositions are immediate consequences of the preceding three lemmas and (3.12).

THEOREM 3.1. *Suppose that* $d_n \ne -1$ *for* $n \ge 1$. *Then:* (i) *if only a finite number of terms of* $\{d_n\}$ $(n \ge 1)$ *are zero and* (3.5) *is satisfied, then* $[F, d_n]$ *is regular if and only if* (3.6) *is satisfied;* (ii) *if an infinite number of terms of* $\{d_n\}$ *are zero, then* $[F, d_n]$ *is regular if and only if, for the coefficient set* $\{p'_{nm}\}$ *belonging to* $[F, d'_n]$ (*where* $\{d'_n\}$ *is the sequence of the nonzero terms of* $\{d_n\}$ *arranged in the same order as in* $\{d_n\}$), *we have*

$$[\,|1 + d'_n|\,!]^{-1} \sum_{m=0}^{n} |p'_{nm}| < K < +\infty \qquad \text{for } n \ge n_0 \ge 1. \tag{3.13}$$

THEOREM 3.2. *If* $\{d_n\}$ $(n \ge 1)$ *is a sequence with* $d_n \ne -1$ *for* $n \ge 1$ *and* $d_n \ge 0$ *for* $n \ge n_0$, *then* $[F, d_n]$ *is regular if and only if*

$$\sum_{m=1}^{\infty}{}' \; 1/d_m = +\infty, \tag{3.14}$$

where the summation is taken over all the nonzero d_m's, *in case there are infinitely many.*

A consequence of Theorem 3.2 is that the $[F, d_n = n - 1]$ (or the L) transformation is regular. This result is due to Lototsky, and was proved by him in [5] and by R. P. Agnew in [2].

4. POWER SERIES

In the application to power series, a certain subclass of the family of $[F, d_n]$ transformations is both potent and simple. We begin with the geometric series $1 + z + z^2 + \cdots$, where z is a complex number $(z \ne 1)$.

THEOREM 4.1. *Suppose that, for the real sequence* $\{d_n\}$ $(n \ge 1)$, $\lim_{n\to\infty} d_n = +\infty$, $d_n \ne -1$ *for* $n \ge 1$ *and* $\Sigma\, d_n^{-2} < +\infty$. *If the* $[F, d_n]$ *transformation is regular, then it sums the series* $1 + z + z^2 + \cdots$ *to* $1/(1 - z)$ *for all complex* z *with* $\Re z < 1$, *and uniformly in each bounded domain inside the half-plane* $\Re z < 1$. *It does not sum* $1 + z + z^2 + \cdots$ *at all points* z *with* $\Re z > 1$.

Proof. For the series $1 + z + z^2 + \cdots$, we have

$$s_n = (1 - z)^{-1} - z^{n+1}(1 - z)^{-1}.$$

Therefore, for the $[F, d_n]$ transform $\{t_n\}$,

$$t_n = [(1+d_n)!]^{-1} \sum_{m=0}^{n} p_{nm} s_m = (1-z)^{-1} - \frac{z}{1-z} \frac{(z+d_n)!}{(1+d_n)!}$$

$$= (1-z)^{-1} - \frac{z}{1-z} \left(1 + \frac{z-1}{1+d_n}\right)! \qquad (n > 0).$$

There is an n_0 such that $d_n > 1$ for all $n \geq n_0$. From the condition $\Sigma_{m=n}^{\infty} d_{n_0}^{-2} < +\infty$ we obtain (see [6, p. 224, Theorem 10 and a slight improvement of it]) that there is a constant $c \equiv c(z) \neq 0$ such that

$$\prod_{m=n_0}^{n} \left(1 + \frac{z-1}{1+d_m}\right) \sim c \exp\left\{(z-1) \sum_{m=n_0}^{n} (1+d_m)^{-1}\right\}$$

$$= c \exp\left\{(x-1) \sum_{m=n_0}^{n} (1+d_m)^{-1}\right\} \cdot \exp\left\{iy \sum_{m=n_0}^{n} (1+d_m)^{-1}\right\}$$

if $z = x + iy$. Hence

$$\left| \prod_{m=n_0}^{n} \left(1 + \frac{z-1}{1+d_m}\right) \right| \sim |c| \exp\left\{(x-1) \sum_{m=n_0}^{n} (1+d_m)^{-1}\right\} \to \begin{cases} 0 & \text{if } x < 1, \\ +\infty & \text{if } x > 1. \end{cases}$$

A simple additional argument completes the proof.

The following proposition is an improvement over Theorem 4.1.

THEOREM 4.2. *Suppose that, for the real sequence* $\{d_n\}$ $(n \geq 1)$, $\lim_{n\to\infty} d_n = +\infty$, $d_n \neq -1$ *for* $n \geq 1$, *and*

$$\sum_{m=1}^{\infty} d_m^{-(p+1)} < +\infty$$

for some nonnegative integer p. *If the* $[F, d_n]$ *transformation is regular, then it sums the series* $1 + z + z^2 + \cdots$ *to* $(1-z)^{-1}$ *for all complex* z *with* $\Re z < 1$, *and uniformly in each bounded domain inside the half-plane* $\Re z < 1$. *It does not sum* $1 + z + z^2 + \cdots$ *at all the points* z *with* $\Re z > 1$.

In the proof of Theorem 4.2, we need the following two lemmas.

LEMMA 4.1. *If* $\{d_n\}$ $(n \geq 1)$ *is a real sequence with* $d_n > 0$ *for* $n \geq n_0$, $\lim_{n\to\infty} d_n = +\infty$ *and*

$$\sum_{m=n_0}^{\infty} d_m^{-1} = +\infty,$$

then, for each fixed integer p $(p > 1)$,

$$\sum_{m=n_0}^{n} d_n^{-p} = o\left(\sum_{m=n_0}^{n} d_m^{-1}\right) \textit{ as } n \to \infty.$$

Lemma 4.1 is a consequence of a well-known proposition (see [7, page 11, paragraph 1.72]).

LEMMA 4.2. *If* $\{d_n\}$ $(n \geq 1)$ *is a real sequence with* $\lim_{n\to\infty} d_n = +\infty$, *then for each positive integer* p *the two series* $\Sigma\, d_m^{-p}$ *and* $\Sigma\,(1 + d_m)^{-p}$ *converge or diverge together.*

Proof of Theorem 4.2. As in the proof of Theorem 4.1, we have

$$t_n = (1 - z)^{-1} - \frac{z}{1 - z}\left(1 + \frac{z - 1}{1 + d_n}\right)!\,.$$

Suppose that $d_n > 1$ for $n \geq n_0$. We can write

$$\prod_{m=n_0}^{n}\left(1 + \frac{z - 1}{1 + d_n}\right) = \prod_{m=n_0}^{n}\left\{\left(1 + \frac{z - 1}{1 + d_m}\right)\exp\left[-\frac{z - 1}{1 + d_m} + \cdots + \frac{(-1)^p}{p}\left(\frac{z - 1}{1 + d_m}\right)^p\right]\right\}$$

$$\cdot \prod_{m=n_0}^{n} \exp\left\{\frac{z - 1}{1 + d_m} - \cdots - \frac{(-1)^p}{p}\left(\frac{z - 1}{1 + d_m}\right)^p\right\}$$

$$= \exp\left\{(z - 1)\sum_{m=n_0}^{n}(d_m + 1)^{-1} - \cdots - \frac{(-1)^p}{p}(z - 1)^p \sum_{m=n_0}^{n}(1 + d_m)^{-p}\right\}$$

$$\cdot \prod_{m=n_0}^{n}\left\{\left(1 + \frac{z - 1}{1 + d_m}\right)\exp\left[-\frac{z - 1}{1 + d_m} + \cdots + \frac{(-1)^p}{p}\left(\frac{z - 1}{1 + d_m}\right)^p\right]\right\}.$$

Now (see [3, p. 130]), as $n \to \infty$, the second factor of this product converges (uniformly for all z belonging to any fixed bounded domain) to an integral function. In the first factor of the product, all the terms in the exponent are small compared with the first term $(z - 1)\Sigma_{m=n_0}^{n}(d_m + 1)^{-1}$ (uniformly for all z belonging to any fixed and bounded domain). The argument in the proof of Theorem 4.1 now completes the proof.

By combining Cauchy's integral formula for analytic functions and Theorem 4.2, we obtain the following proposition.

THEOREM 4.3. *Suppose that, for the real sequence* $\{d_n\}$ $(n \geq 1)$, $d_n \neq -1$ *for* $n \geq 1$, $\lim_{n\to\infty} d_n = +\infty$, $\Sigma\, d_m^{-(p+1)} < +\infty$ *for some nonnegative integer* p, *and the* $[F, d_n]$ *transformation is regular. If* $\Sigma_{n=0}^{\infty} a_n z^n$ *is a power series with a positive radius of convergence, and if* F(z) *is the analytic function which is generated by analytic extension along radial lines from the origin of the complex plane, then* $\Sigma\, a_n z^n$ *is summable* $[F, d_n]$ *to* F(z) *at each point inside the Borel polygon of* F(z).

5. RELATION BETWEEN THE $[F, d_n]$ TRANSFORMATIONS AND THE EULER-KNOPP TRANSFORMATIONS

In this section we shall obtain inclusion relations between a certain class of $[F, d_n]$ transformations and the [E, q] transformations. It is known that if $\{e_n^{(q)}\}$ is the [E, q] transform of $\{s_n\}$, then $\{s_n\}$ is the $[E, q^{-1}]$ transform of $\{e_n^{(q)}\}$. Our first result:

THEOREM 5.1. *Suppose* $\{d_n\}$ $(n \geq 1)$ *is a sequence with* $d_n \neq -1$ *for* $n \geq 1$. *Then the* $[F, d_n]$ *transformation includes the* $[E, q]$ *transformation* $(q \neq 0$; q *a real or complex number*) *if and only if the transformation* $[F, d_n' = q(d_n + 1) - 1]$ *is regular.*

The following two propositions are consequences of Theorem 5.1 and the results of Section 3.

THEOREM 5.2. *Suppose that, for the real sequence* $\{d_n\}$ $(n \geq 1)$, $d_n \neq -1$ *for* $n \geq 1$ *and* $d_n > 0$ *for* $n \geq n_0$. *If* $[F, d_n]$ *is regular, then for each real* q *such that* $q(d_n + 1) \geq 1$ *for all* $n \geq n_2(q)$, $[F, d_n]$ *includes* $[E, q]$.

THEOREM 5.3. *Suppose that* $\{d_n\}$ $(n \geq 1)$ *is a real sequence with* $d_n \neq -1$ *for* $n \geq 1$ *and* $\lim_{n\to\infty} d_n = +\infty$. *If* $[F, d_n]$ *is regular, then it includes all the* $[E, q]$ *transformations with* $q > 0$.

In the proof of Theorem 5.1 we use the following lemma.

LEMMA 5.1. *The* $[F, d_n]$ *transform* $\{t_n\}$ *of* $\{s_n\}$ *is the* $[F, d_n' = q(d_n + 1) - 1]$ *transform of* $\{e_n^{(q)}\}$ $(q \neq 0)$.

Proof. Denote by E the 'displacement operator' applicable to sequences; that is, let

$$Eu_n = u_{n+1}; \qquad E^0 u_n = u_n; \qquad E^{n+1} = E(E^n).$$

Now, for $q \neq 0$,

$$e_n^{(q)} = (1 - q + qE)^n s_0,$$

$$t_n = [(1 + d_n)!]^{-1} (E + d_n)!\, s_0,$$

$$s_n = \left(1 - \frac{1}{q} + \frac{1}{q} E^*\right)^n e_0^{(q)}.$$

Hence

$$t_n = [(1 + d_n)!]^{-1} \left(1 - \frac{1}{q} + \frac{1}{q} E^* + d_n\right)!\, e_0^{(q)}$$

$$= \{[q(1 + d_n)]!\}^{-1} [E^* + q(d_n + 1) - 1]!\, e_0^{(q)},$$

Q. E. D. Theorem 5.1 now follows immediately from Lemma 5.1, since the $[E, q]$ transformation has an inverse for $q \neq 0$. We can improve Theorem 5.3 as follows,

THEOREM 5.4. *Suppose that* $\{d_n\}$ $(n \geq 1)$ *is a real sequence with* $d_n \neq -1$ *for* $n \geq 1$ *and* $\lim_{n\to\infty} d_n = +\infty$. *If the* $[F, d_n]$ *transformation is regular, then it includes the* $[E, q]$ *transformation if and only if* $q > 0$.

The particular case $d_n = n - 1$ for $n \geq 1$ was proved in [2, Section 8]. In the proof of the theorem, we need the following lemma.

LEMMA 5.2. *Suppose that* $\{d_n\}$ $(n \geq 1)$ *is a real sequence with* $d_n \neq -1$ *for* $n \geq 1$, $\lim_{n\to\infty} d_n = +\infty$ *and* $\Sigma d_n^{-1} = +\infty$. *Then the* $[F, d_n' = q(d_n + 1) - 1]$ *transformation is not regular for all* $q < 0$.

Proof. For $q < 0$, we write $q = -p$, so that $p > 0$. If

$$\prod_{m=1}^{n} [x + q(d_m + 1) - 1] = \prod_{m=1}^{n} [x - (pd_m + p + 1)] \equiv \sum_{m=0}^{n} p_{nm} x^m$$

and

$$\prod_{m=1}^{n} [x + (pd_m + p + 1)] = \sum_{m=0}^{n} p^*_{nm} x^m ,$$

then

$$\sum_{m=0}^{n} |p_{nm}| = \sum_{m=0}^{n} |p^*_{nm}| .$$

Now, if $d_n > 0$ for $n \geq n_0$, then

$$\left|\frac{(1 + pd_n + p + 1)!}{(1 - pd_n - p - 1)!}\right| = \prod_{m=1}^{n_0} \left|\frac{p(d_m + 1) + 2}{p(d_m + 1)}\right| \cdot \prod_{m=n_0+1}^{n} \left[1 + \frac{2}{p(d_m + 1)} \right].$$

Therefore

$$\lim_{n \to \infty} \left|\frac{(1 + pd_n + p + 1)!}{(1 - pd_n - p - 1)!}\right| = +\infty .$$

By Theorem 3.2, the $[F, d'_n = p(d_n + 1) + 1]$ transformation is regular, and therefore

$$0 < C \leq \{[1 + p(d_n + 1) + 1]!\}^{-1} \sum_{m=0}^{n} |p^*_{nm}| < K < +\infty \qquad \text{for } n \geq n_1 .$$

Hence

$$\frac{\sum_{m=0}^{n} |p_{nm}|}{\left|\prod_{m=1}^{n} [1 + q(d_m + 1) - 1]\right|} = \frac{\sum_{m=0}^{n} |p^*_{nm}|}{\left|\prod_{m=1}^{n} [1 + p(d_m + 1) + 1]\right|} \left|\frac{\prod_{m=1}^{n} [1 + p(d_m + 1) + 1]}{\prod_{m=1}^{n} [1 + q(d_m + 1) - 1]}\right| \to +\infty$$

as $n \to \infty$.

Thus the $[F, d'_n = q(d_n + 1) - 1]$ transformation is not regular. *Q. E. D.*

Proof of Theorem 5.4. By Theorem 5.3, the $[F, d_n]$ transformation includes the $[E, q]$ transformation for each $q > 0$. By Lemma 5.2, $[F, d_n]$ does not include $[E, q]$ for each $q < 0$. If q is a complex number, then the $[E, q]$ transformation sums the power series $1 + z + z^2 + \cdots$ inside the circle with center at $1 - 1/q$ and passing through the point 1 (see [2, p. 121]). The interior of this circle includes points z with $\Re z > 1$. On the other hand, by Theorem 4.1, the $[F, d_n]$ transformation does not sum $1 + z + z^2 + \cdots$ at all points z with $\Re z > 1$. Therefore $[F, d_n]$ does not include [E, q] for complex q. The [E, 0] transform $\{e_n^{(0)}\}$ of $\{s_n\}$ satisfies $e_n^{(0)} = s_0$ for $n \geq 0$. Therefore each sequence is summable [E, 0]; but there are sequences which are not summable $[F, d_n]$ (for example, $1 + z + z^2 + \cdots$ for $\Re z > 1$); thus $[F, d_n]$ does not include [E, 0]. *Q. E. D.*

6. SERIES-TO-SERIES VERSION OF $[F, d_n]$

Suppose that $t_0, t_1, \cdots$ is the $[F, d_n]$ transform of the sequence $s_0, s_1, \cdots$. Let

$$s_n = \sigma_0 + \sigma_1 + \cdots + \sigma_n, \tag{6.1}$$

$$t_n = \tau_0 + \tau_1 + \cdots + \tau_n. \tag{6.2}$$

We see that

$$\tau_0 = t_0 = s_0 = \sigma_0. \tag{6.3}$$

For $n > 0$, we have $\tau_n = t_n - t_{n-1}$ and hence, for $n > 0$,

$$\tau_n = \sum_{k=0}^{n} \left[\frac{p_{nk}}{(1 + d_n)!} - \frac{p_{n-1,k}}{(1 + d_{n-1})!}\right] \sum_{j=0}^{k} \sigma_j$$

$$= \frac{1}{(1 + d_n)!} \sum_{j=0}^{n} \left\{\sum_{k=j}^{n} [p_{nk} - (1 + d_n)p_{n-1,k}]\right\} \sigma_j,$$

and by (1.2),

$$\tau_n = \frac{1}{(1 + d_n)!} \sum_{j=0}^{n} \left\{\sum_{k=j}^{n} [p_{n-1,k-1} - p_{n-1,k}]\right\} \sigma_j.$$

Thus

$$\tau_n = \frac{1}{(1 + d_n)!} \sum_{j=1}^{n} p_{n-1,j-1}\, \sigma_j = \frac{1}{(1 + d_n)!} \sum_{m=0}^{n} p_{n-1,m}\sigma_{m+1}. \tag{6.4}$$

From (6.1), (6.2), (6.3) and (6.4) we see that $\Sigma_{m=0}^{\infty} \sigma_m$ is summable $[F, d_n]$ to s if and only if $\Sigma_{m=0}^{\infty} \tau_m$ is convergent to s. When the series $\sigma_0 + \sigma_1 + \sigma_2 + \cdots$ is summable $[F, d_n]$, we may denote the sum by the left member of the formula

$$[F, d_n]\left\{\sum_{m=0}^{\infty} \sigma_m\right\} = \sigma_0 + \sum_{n=1}^{\infty} \frac{1}{(1 + d_n)!} \sum_{k=0}^{n-1} p_{n-1,k}\sigma_{k+1}. \tag{6.5}$$

The particular case $[F, d_n] \equiv L$ of (6.5) was proved in [2, Section 6].

7. THE SEQUENCE $0, s_0, s_1, \cdots$

We shall see that for a certain class of $[F, d_n]$ transformations the relation between the $[F, d_n]$ transforms of the sequence $0, s_0, s_1, \cdots$ and $s_0, s_1, s_2, \cdots$ is quite simple.

Denote by $\{t_n\}$ the $[F, d_n]$ transform of $s_0, s_1, s_2, \cdots$, and by $\{t_n^{(1)}\}$ the $[F, d_n]$ transform of $0, s_0, s_1, \cdots$. Multiplying the recursion formula (see (1.2))

$$p_{n+1,k} = d_{n+1}\, p_{nk} - p_{n,k-1}$$

by $\dfrac{s_{k-1}}{(1 + d_n)!}$, and summing over $1 \leq k \leq n + 1$, we obtain

$$(1 + d_{n+1}) \sum_{k=1}^{n+1} \frac{p_{n+1,k}\, s_{k-1}}{(1 + d_{n+1})!} = d_{n+1} \sum_{k=1}^{n+1} \frac{p_{nk}\, s_{k-1}}{(1 + d_n)!} + \sum_{k=1}^{n+1} \frac{p_{n,k-1}\, s_{k-1}}{(1 + d_n)!},$$

and hence

$$(7.1) \qquad (1 + d_{n+1})\, t_{n+1}^{(1)} = d_{n+1}\, t_n^{(1)} + t_n .$$

From (7.1) we obtain

$$(7.2) \qquad t_{n+1}^{(1)} = \frac{1}{1 + d_{n+1}} \cdot \left[\frac{d_{n+1} \cdots d_2}{(1 + d_n) \cdots (1 + d_1)}\, t_0 + \cdots + \frac{d_{n+1}}{1 + d_n}\, t_{n-1} + t_n \right].$$

It is easy to see that

$$(7.3) \qquad 1 + d_{n+1} = \frac{d_{n+1} \cdots d_1}{(1 + d_n) \cdots (1 + d_1)} + \left[\frac{d_{n+1} \cdots d_2}{(1 + d_n) \cdots (1 + d_1)} + \cdots + \frac{d_{n+1}}{1 + d_n} + 1 \right].$$

Suppose now that $d_n \geq 0$ for $n \geq 1$. It is easy to see by (7.3) that, in the case $d_n \geq 0$ for $n \geq 1$, the transformation from $\{t_n\}$ to $t_{n+1}^{(1)}$ defined by (7.2) is regular if and only if

$$\prod_{m=1}^{\infty}{}' \left(1 + \frac{1}{d_m} \right) = +\infty,$$

where Π' denotes the product of all terms $(1 + 1/d_m)$ with $d_m \neq 0$ if there is only a finite number of zero terms in $\{d_n\}$. Otherwise it is always regular. By Theorem 3.2 and the consideration above, we obtain the following proposition.

THEOREM 7.1. *Suppose that* $\{d_n\}$ $(n \geq 1)$ *is a real sequence with* $d_n \geq 0$ *for* $n \geq n_0$. *Then* (i) *if* $[F, d_n]$ *is regular and there are only a finite number of zero terms in* $\{d_n\}$, *then the* $[F, d_n]$ *summability of* $s_0, s_1, \cdots$ *implies the* $[F, d_n]$ *summability of* $0, s_0, s_1, \cdots$ *to the same sum.* (ii) *If there are infinitely many zero terms in* $\{d_n\}$, *then the* $[F, d_n]$ *summability of* $s_0, s_1, \cdots$ *implies the* $[F, d_n]$ *summability of* $0, s_0, s_1, \cdots$ *to the same sum.*

The particular case $[F, d_n] \equiv L$ of Theorem 7.1 was proved in [2, Section 7].

8. THE INVERSE OF THE $[F, d_n]$ TRANSFORMATION

If $f(x)$ is a function defined at least for $x = x_1, x_2, \cdots$, and if all the x_v are different, we define the *divided differences* of $f(x)$ by means of the relations

$$(8.1)\quad \begin{cases} [f(x_v)] = f(x_v), \\ [f(x_v), f(x_{v+1})] = \dfrac{[f(x_v)] - [f(x_{v+1})]}{x_v - x_{v+1}}, \\ [f(x_v), \cdots, f(x_{u+1})] = \dfrac{[f(x_v), \cdots, f(x_u)] - [f(x_{v+1}), \cdots, f(x_{u+1})]}{x_v - x_{u+1}} \qquad (u \geq v). \end{cases}$$

By Newton's expansion, each polynomial $f(x)$ satisfies the relation

$$(8.2)\qquad f(x) = [f(x_1)] + (x - x_1)[f(x_1), f(x_2)] + (x - x_1)(x - x_2)[f(x_1), \cdots, f(x_3)] + \cdots$$

(see [4, Section 23, p. 74]).

Suppose that $x_1, x_2, \cdots, x_n$ are real and lie on the closed interval $<a, b>$. It is known (see [8, Section 2.7]) that if $f(z)$ is a regular analytic function at each point of $<a, b>$, then, for $1 \leq k \leq n$, there is a ξ such that $a \leq \xi \leq b$ and

$$(8.3)\qquad [f(x_1), \cdots, f(x_k)] = \frac{1}{(k-1)!} f^{(k-1)}(\xi).$$

In particular, $[x_1^m, \cdots, x_k^m] = 0$ if $k > m + 1$.

Choosing $x_v = d_v$ $(v \geq 1)$, where we suppose all the d_v different, we obtain from (8.2)

$$x^n = [d_1^n] + (x - d_1)[d_1^n, d_2^n] + (x - d_1)(x - d_2)[d_1^n, d_2^n, d_3^n] + \cdots.$$

Replacing x by $-x$ in the last identity, we obtain

$$(8.4)\qquad (-1)^n x^n = [d_1^n] - (x + d_1)[d_1^n, d_2^n] + (x + d_1)(x + d_2)[d_1^n, d_2^n, d_3^n] + \cdots.$$

Now write

$$(8.5)\qquad q_{nm} = (-1)^{n-m}[d_1^n, \cdots, d_{m+1}^n].$$

From (8.3) we see that if $d_n \geq 0$ for $n \geq 1$, then

$$(8.6)\qquad (-1)^{n-m} q_{nm} = |q_{nm}| = [d_1^n, \cdots, d_{m+1}^n] \geq 0.$$

If $\{t_n\}$ is the $[F, d_n]$ transform of $\{s_n\}$ and all the $\{d_v\}$ are different, then, by (8.4) and (8.5),

$$(8.7)\qquad s_n = \sum_{m=0}^{n} (1 + d_m)!\, q_{nm}\, t_m \qquad (\text{here } (1 + d_0)! \equiv 1).$$

From (8.7) and (9.6) we obtain

THEOREM 8.1. *Suppose that* $\{d_n\}$ $(n \geq 1)$ *is a real sequence with* $d_n \geq 0$ *for* $n \geq 1$ *and with all the* d_n *distinct. If* $\{s_n\}$ *is summable* $[F, d_n]$, *then*

$$s_n = O\left(d_1^n + \sum_{m=1}^{n} (1 + d_m)!\,[d_1^n, \cdots, d_{m+1}^n]\right) \quad \text{as } n \to \infty. \tag{8.8}$$

As we have seen, $[d_1^n, \cdots, d_m^n] = 0$ if $m > n + 1$ in case $d_v \geq 0$; thus Theorem 8.1 can be stated in the following alternative form.

THEOREM 8.2. *Under the suppositions of Theorem* 8.1,

$$s_n = O\left(d_1^n + \sum_{m=1}^{\infty} (1 + d_m)!\,[d_1^n, \cdots, d_{m+1}^n]\right) \quad \textit{as } n \to \infty. \tag{8.9}$$

In the case $d_v = v - 1$ $(v \geq 1)$, the expression inside the brackets on the right of (8.8) can be simplified in the following way.

$$\begin{aligned} d_1^n + \sum_{m=1}^{n} (1 + d_m)!\,[d_1^n, \cdots, d_{m+1}^n] &= \lim_{x \uparrow 1}\left\{d_1^n + \sum_{m=1}^{n} (1 + d_m)!\,[d_1^n, \cdots, d_{m+1}^n]\,x^m\right\} \\ &= \lim_{x \uparrow 1}\left\{d_1^n + \sum_{m=1}^{\infty} (1 + d_m)!\,[d_1^n, \cdots, d_{m+1}^n]\,x^m\right\} \\ &= \lim_{x \uparrow 1} \sum_{m=0}^{\infty} (-1)^m x^m \sum_{i=0}^{m} (-1)^i \binom{m}{i} i^n. \end{aligned} \tag{8.10}$$

From the inequality $\Sigma_{i=0}^m \binom{m}{i} i^n \leq m^n 2^m$, we see that for $|x| < 1/2$ we may change the order of summation in the last line of (8.10), and we obtain (for $|x| < 1/2$)

$$\begin{aligned} \sum_{m=0}^{\infty} (-1)^m x^m \sum_{i=0}^{m} (-1)^i \binom{m}{i} i^n &= \sum_{i=0}^{\infty} (-1)^i i^n \sum_{m=i}^{\infty} (-1)^m \binom{m}{i} x^m \\ &= \sum_{i=0}^{\infty} i^n \sum_{r=0}^{\infty} (-1)^r \binom{r+i}{r} x^{r+i} = \frac{1}{1+x} \sum_{i=0}^{\infty} i^n (1+x)^{-i}. \end{aligned}$$

By the principle of analytic continuation, we see that

$$d_1^n + \sum_{m=1}^{n} (1 + d_m)!\,[d_1^n, \cdots, d_{m+1}^n] = \frac{1}{1+1} \sum_{i=0}^{\infty} i^n \cdot (1+1)^{-i} = \frac{1}{2} \sum_{i=1}^{\infty} \frac{i^n}{2^i}. \tag{8.11}$$

(8.11) is (5.6) of [2]. But here the proof is direct.

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Added in proof. After this paper had been submitted for publication, I received from Prof. J. Karamata a reprint of his paper *Théorèmes sur la sommabilité exponentielle et d'autres sommabilités s'y rattachant*, Mathematica, Cluj 9 (1935), 164-178. There what we have called the *Lototsky method* was defined (and called the *Stirling method*), together with what in the notation of our paper are the $\left[F, d_n = \frac{1}{k}(n-1)\right]$ methods (k = fixed = 1, 2, $\cdots$). In Prof. Karamata's paper, Abelian and Tauberian inclusion theorems between the Euler, Stirling-Karamata-Lototsky, and Borel methods are proved.

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