ON MINIMAL COMPLETELY REGULAR SPACES ASSOCIATED WITH A GIVEN RING OF CONTINUOUS FUNCTIONS

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1. INTRODUCTION

Let $C(X)$ denote the ring of all continuous real-valued functions on a completely regular space X. If X and Y are completely regular spaces such that one is dense in the other, say X is dense in Y, and every $f \in C(X)$ has a (unique) extension $\bar{f} \in C(Y)$, then $C(X)$ and $C(Y)$ are said to be *strictly isomorphic*. In a recent paper [2], L. J. Heider asks if it is possible to associate with the completely regular space X a dense subspace μX minimal with respect to the property that $C(\mu X)$ and $C(X)$ are *strictly isomorphic*.[1]

In this note, Heider's question is answered in the negative. It is shown, moreover, that if μX exists, then it consists of all of the isolated points of X, together with those nonisolated points p of X such that $C(X \sim \{p\})$ and $C(X)$ fail to be strictly isomorphic. Thus, if μX exists, it is unique.

2. PRELIMINARY REMARKS

Let $C(X)$ denote the ring of all continuous real-valued functions on a completely regular space X. Let $C^*(X)$ denote the subring of all bounded $f \in C(X)$. The following known facts are utilized below.

(2.1) Corresponding to each completely regular space X, there exists an essentially unique compact space βX, called the Stone-Čech compactification of X, such that (i) X is dense in βX, and (ii) every $f \in C^*(X)$ has a (unique) extension $\bar{f} \in C^*(\beta X) = C(\beta X)$. Thus $C^*(X)$ and $C(\beta X)$ are isomorphic. (See, for example, [3] or [4, Chapter 5].)

(2.2) There exists an essentially unique subspace υX of βX such that (i) X is a Q-space, (ii) X is dense in υX, and (iii) every $f \in C(X)$ has a (unique) extension $\bar{f} \in C(\upsilon X)$. Thus $C(X)$ and $C(\upsilon X)$ are isomorphic. (For the definition of Q-space, and a proof of this theorem, see [1] or [3].)

(2.3) If X and Y are completely regular spaces such that $C(X)$ and $C(Y)$ are isomorphic, then Y is homeomorphic to a dense subspace of υX such that every real-valued function continuous on this subspace has a (unique) continuous extension over υX. [3, Theorem 65.]

(2.4) If Z is any compact space, and f is any continuous mapping of X into Z, then there exists a (unique) continuous extension $\hat{f}$ of f over βX into Z. (See [5, Theorem 88].)

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1. Since the writing of this paper, Heider's problem has been generalized and solved independently by J. Daly and L. J. Heider.

In the oral presentation of [2], Heider asked "... whether or not to each completely regular space X, there is associated a completely regular space μX such that μX and $\upsilon(\mu X)$ are homeomorphic, and $\mu X \subset Y \subset \upsilon X$ for every completely regular space Y such that υY is homeomorphic to υX." By considering the special case $Y = X$ in Heider's formulation, we see at once that $\mu X \subset X$. Moreover, since $\upsilon(\mu X)$ and υX are homeomorphic, it follows from (2.3) that μX is homeomorphic to a dense subspace of X all of whose continuous real-valued functions have continuous extensions over X. Thus, it is natural to identify μX with its image in X under this homeomorphism; this identification leads to the formulation of Heider's problem given in the Introduction, namely: does there exist a dense subspace μX of X which is minimal with respect to the property that $C(\mu X)$ and $C(X)$ are strictly isomorphic?[1]

We conclude this section with a definition.

Definition. If X is a completely regular space, let ηX denote the union of the set of isolated points of X and the set of nonisolated points p of X such that $C(X \sim \{p\})$ and $C(X)$ fail to be strictly isomorphic.

Thus, by (2.3), a nonisolated point p of X fails to be in ηX if and only if every $f \in C(X \sim \{p\})$ has a (unique) continuous extension over X.

3. UNIQUENESS OF μX

We begin this section with a theorem which will be used below, and which we believe to be of some independent interest.

THEOREM 3.1. *If* Y *is a dense subspace of a completely regular* X *such that the rings* $C(Y)$ *and* $C(X)$ *(respectively,* $C^*(Y)$ *and* $C^*(X)$*) are strictly isomorphic, then, for any (nonisolated) point* $p \in Y$, *the rings* $C(Y \sim \{p\})$ *(respectively,* $C^*(Y \sim \{p\})$ *and* $C^*(X \sim \{p\})$*) are strictly isomorphic.*

Proof. Except for the part of the theorem in parentheses, it is enough, by (2.3), to show that every $f \in C(Y \sim \{p\})$ has a (unique) extension $F \in C(X \sim \{p\})$. As for the part in parentheses, it will be evident from the construction that if $f \in C^*(Y \sim \{p\})$, then $F \in C^*(X \sim \{p\})$.

Let $\{U_\alpha\}_{\alpha \in A}$ be a base of neighborhoods in the space X of p. The index set A becomes a directed set if we let the statement $\beta \geq \alpha$ mean that $U_\beta \subset U_\alpha$. Since X is completely regular, for each $\alpha \in A$, there exists an $i_\alpha \in C^*(X)$ such that $i_\alpha(x) = 1$ for $x \in X \sim U_\alpha$, and i_α vanishes on a neighborhood of p. (To see this, let $h_\alpha \in C^*(X)$ be such that $h_\alpha(X \sim U_\alpha) = 1$, and $h_\alpha(p) = -1$. Then let $i_\alpha(x) = \max(h_\alpha(x), 0)$ for every $x \in X$.) Let f be the function defined on Y by letting $f_\alpha(y) = i_\alpha(y) f(y)$ for every $y \in Y \sim \{p\}$, and by letting $f_\alpha(p) = 0$. Clearly, $f_\alpha \in C(Y)$, and $f_\alpha(y) = f(y)$ for all y outside of U_α. Now, by hypothesis (and (2.3)), f_α has a unique extension $F_\alpha \in C(X)$.

For each $x \in X \sim \{p\}$, the set $\{F_\alpha(x)\}_{\alpha \in A}$ forms a real-valued net [4, Chapter 2]. For each $x \in X \sim \{p\}$, let $F(x) = \lim_\alpha F_\alpha(x)$. This limit exists since, if U_{α_x} is a basic neighborhood of p disjoint from x, it follows from $\beta \geq \alpha_x$ that

$$F_{\alpha_x}(x) = F_\beta(x) = F(x).$$

It is clear that F is an extension of f. We will show next that $F \in C(X \sim \{p\})$, by verifying that F is continuous at each $x_0 \in X \sim \{p\}$.

Let V_{x_0}, U_{α_0} denote disjoint neighborhoods (in X) respectively of x_0 and p. If $x \in V_{x_0}$, then for any $\beta \geq \alpha_0$, $F(x) = F_\beta(x)$. Hence the continuity of F at x_0 follows from the continuity of F_β at x_0. This completes the proof of the theorem.

COROLLARY. *If* Y *is a dense subspace of the completely regular space* X *then, for any (nonisolated) point* $p \in Y$, *if* υY *and* υX *(respectively,* βY *and* βX) *are homeomorphic, then* $\upsilon(Y \sim \{p\})$ *and* $\upsilon(X \sim \{p\})$ *(respectively,* $\beta(Y \sim \{p\})$ *and* $\beta(X \sim \{p\})$) *are homeomorphic.*

It will be shown next that if μX exists, then it is unique.

THEOREM 3.2. *If with the completely regular space* X *there is associated at least one dense subspace* μX *minimal with respect to the property that* $C(\mu X)$ *and* $C(X)$ *are strictly isomorphic, then* μX *is unique. In fact,* $\mu X = \eta X$.

Proof. It follows from the definition of ηX, and from the fact that μX is dense in X, that each of these spaces contains all the isolated points of X. Hence we need only consider the nonisolated points of X. We will show first that $\mu X \subset \eta X$.

Let p be a nonisolated point of X contained in μX. By the minimality of μX, there exists an $f \in C(\mu X \sim \{p\})$ with no continuous extension over μX. But, by Theorem 3.1, f has an extension $F \in C(X \sim \{p\})$. If p were not in ηX, F would have a continuous extension over X, whose restriction to μX would in turn be a continuous extension of f over μX. Hence $p \in \eta X$, whence $\mu X \subset \eta X$.

Suppose there were a point $p \in \eta X \sim \mu X$. If $f \in C(X \sim \{p\})$, then since $C(\mu X)$ and $C(x)$ are isomorphic, the restriction of f to μX has a continuous extension over X. This latter would be a continuous extension of f over X, contrary to the assumption that $p \in \eta X$. Hence $\mu X = \eta X$. This completes the proof of the theorem.

COROLLARY. *A necessary and sufficient condition that* μX *exist (in which case it is equal to* ηX) *is that* ηX *be dense in* X *and that every* $f \in C(\eta X)$ *have a (unique) extension* $\bar{f} \in C(X)$.

As noted by Heider [2], $\mu X = \eta X = X$, provided every point of X is a G_δ.

4. THE SUBSPACE μX NEED NOT EXIST

In this section we give an example of a completely regular space X such that μX does not exist. In fact, for this X, ηX is dense in X, but $C(\eta X)$ and $C(X)$ are not isomorphic.

We begin by generalizing a result of Hewitt [3, p. 62].

THEOREM 4.1. *Let* Y *be a noncompact completely regular space, and suppose that* $Y \subset X \subset \beta Y$ *and that* $\beta Y \sim X$ *has power less than* $\exp \exp \aleph_0$. *Then* $\upsilon X = \beta X = \beta Y$. *In particular,* $C(X) = C^*(X)$.

Proof. We will show first that $C(X) = C^*(X)$, thus verifying that $\upsilon X = \beta X$. (See (2.1) and (2.2).) For any $f \in C(X)$, let f^* denote its restriction to Y. As noted in [1], f^* may be regarded as a continuous mapping of Y into the one-point compactification $R \cup \{\infty\}$ of the real line R. By (2.4), f^* has a (unique) continuous extension $\hat{f}^*$ over βY into $R \cup \{\infty\}$. Since Y is dense in X, the function $\hat{f}^*$ is also an extension of f. Now the set $G = \{y \in Y: \hat{f}^*(y) = \infty\}$ is a closed G_δ of βY, and it is contained in $\beta Y \sim X \subset \beta Y \sim Y$. Hewitt has shown [3, Theorem 49] that every nonempty closed G_δ of βY contained in $\beta Y \sim Y$ has power at least $\exp \exp \aleph_0$. On the other hand it is evident, from the hypothesis, that G has power less than $\exp \exp \aleph_0$. Hence G is empty. So $f^* \in C^*(Y)$, and it follows that $f \in C^*(X)$. Thus $\upsilon X = \beta X$.

Since X is dense in βY, and βY is compact, in order to conclude that $\beta X = \beta Y$ it suffices, by (2.1), to show that every $f \in C^*(X)$ has a (unique) extension $\bar{f} \in C^*(\beta X)$. We may take $\bar{f}$ to be the (unique) extension over βY of the restriction of f to Y. This completes the proof of the theorem.

Example. Let Y be any completely regular space that admits unbounded continuous real-valued functions, and such that $\eta Y = Y$. (In particular, Y could be any infinite discrete space.) Let $X = \beta Y$. For each $p \in \beta Y \sim Y$, it follows from Theorem 4.1 that $\upsilon(X \sim \{p\}) = X$. Hence, $\eta X \subset Y$, and since $\eta Y = Y$, it follows that $\eta X = Y$. But, although ηX is dense in X, no unbounded $f \in C(Y)$ has a continuous extension over the compact space X. Thus, by the corollary to Theorem 3.1, μX does not exist.

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