A THEOREM ABOUT LOCAL BETTI GROUPS

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SUMMARY. The theory of exact sequences is applied to local homology invariants to obtain a relation analogous to the Mayer-Vietoris formula. This formula is then applied to the local Betti numbers in an lc^n space, and it yields a result concerning generalized manifolds with boundary; this result is described in Section 2.

1. The local Betti numbers are basic tools in the theory of generalized manifolds (see Wilder [8], p. 190 ff.) Alexandroff discussed these invariants in an earlier paper, calling them *Betti numbers around a point* ([1], p. 2). In the same paper, the *Betti groups in a point* were defined (p. 19), and the equivalence of r-dimensional local Betti numbers and the rank of the r^{th} Betti group in a point where there is no r-condensation was proved (Corollary to Theorem III, p. 22). Wilder has proved [6] that no point has r-condensation in an lc^n space, for $r \leq n$, and he has recently shown [7] that Alexandroff's equivalence result holds in a much wider class of spaces. In the following, the theorems are stated and proved for Betti groups in a point, and they are then applied to concepts defined in terms of local Betti numbers, the rank of the r^{th} Betti group being equal to the r-dimensional local Betti number when such application is made.

Let S be a locally compact space, X a closed subset of S, A a closed subset of X, and x a point in A. In analogy with the terminology in the large, (x: X, A) is called a local pair. Let $\{P_\alpha\}$ be a basis for the open sets of S containing x. Denote by $Z^n(x: X, X - P_\alpha)$ the vector space of Čech n-cycles on X mod $(X - P_\alpha)$, with coefficients in a field; denote by $B^n(x: X, X - P_\alpha)$ the subspace of $Z^n(x: X, X - P_\alpha)$ which consists of n-cycles that bound on X mod $(X - P_\alpha)$. The quotient of these spaces is denoted by $H^n(x: X, X - P_\alpha)$. Inclusion among sets in the collection $\{P_\alpha\}$ induces an order relation among the indices $\{\alpha\}$; that is, $P_\alpha \subset P_\beta$ implies $\alpha > \beta$, and the direct limit group $\lim_{\rightarrow} H^n(x: X, X - P_\alpha)$ can be defined. This is called the *Betti group in a point*, and it will be denoted $LH^n(x: X)$.

For the study of the Betti groups in a point of a local pair, it is convenient to consider further the limit groups

$$\lim_{\rightarrow} H^n(x: A \cup (X - P_\alpha), X - P_\alpha) = LH^n(x: A),$$

$$\lim_{\rightarrow} H^n(x: X, A \cup (X - P_\alpha)) = LH^n(x: X, A)$$

and the maps

$$i_*: LH^n(x: A) \rightarrow LH^n(x: X),$$

$$j_*: LH^n(x: X) \rightarrow LH^n(x: X, A),$$

$$\partial : LH^n(x: X, A) \rightarrow LH^{n-1}(x: A),$$

which are defined as the maps of the limit groups defined by the inclusion maps and boundaries of the terms.

Received March 27, 1956.

THEOREM 1. *The sequence of groups formed by the Betti groups in a point of a local pair* $(x: X, A)$,

$$\cdots \to LH^n(x: A) \to LH^n(x: X) \to LH^n(x: X, A) \to LH^{n-1}(x: A) \to \cdots,$$

is exact.

Proof. The sequence of Čech homology groups

$$\cdots \to H^n(x: A \cup (X - P_\alpha), X - P_\alpha) \to H^n(x: X, X - P_\alpha) \to$$
$$H^n(x: X, A \cup (X - P_\alpha)) \to H^{n-1}(x: A \cup (X - P_\alpha), X - P_\alpha) \to \cdots$$

is exact. A direct limit sequence is formed naturally from the definition of Betti groups in a point; the direct limit of exact sequences is exact ([2], p. 225). This is the desired result.

Definition. A *local triad* $(x: X; X_1, X_2)$ consists of a point x, a space X, and two closed subsets X_1, X_2 of X, each of which contains x. A local triad is called *proper* if the inclusion maps

$$k_2: (x: X_1, X_1 \cap X_2) \to (x: X_1 \cup X_2, X_2),$$

$$k_1: (x: X_2, X_1 \cap X_2) \to (x: X_1 \cup X_2, X_1)$$

induce isomorphisms of the homology groups in all dimensions.

The Mayer-Vietoris sequence of homology groups in a point is the sequence

$$\cdots \to LH^n(x: A) \overset{\psi}{\to} LH^n(x: X_1) + LH^n(x: X_2) \overset{\phi}{\to} LH^n(x: X) \overset{\Delta}{\to} LH^{n-1}(x: A) \to \cdots,$$

where $X = X_1 \cup X_2$, $A = X_1 \cap X_2$, and the maps ψ, ϕ, Δ, are defined by

$$\psi u = (h_{1*}u, -h_{2*}u) \qquad (u \in LH^n(x: A)),$$

$$\phi(v_1, v_2) = m_{1*}v_1 + m_{2*}v_2 \qquad (v_1 \in LH^n(x: X_1),\ v_2 \in LH^n(x: X_2)),$$

$$\Delta w = -\partial_1 k_{1*} l_{1*} w = \partial_2 k_{2*} l_{2*} w \qquad (w \in LH^n(x: X)),$$

the maps h_1, h_2, m_1, m_2, l_1, l_2, k_1, k_2 being inclusions of the local pairs, entirely analogous to the inclusions defined in Eilenberg-Steenrod ([2], p. 34 ff.), and the maps h_{1*}, h_{2*}, $\cdots$, being the associated maps for the Betti groups in a point.

THEOREM 2. *The Mayer-Vietoris sequence of homology groups in a point of a proper local triad is exact.*

Proof. In Eilenberg-Steenrod ([2], p. 39), the proof of Theorem 15.3 depends only on the Axioms 1 to 4 for homology theory. These axioms are satisfied by the groups $LH^n(x: X)$ and the corresponding mappings. The proof of Theorem 15.3 can be used without change to prove this theorem.

Definition. Let $\zeta_\nu: LH^q(x: A) \to LH^q(x: X_\nu)$ $(\nu = 1, 2)$ be the maps induced by the inclusion maps, and define

$$r(x: N^q) = \text{rank}(\ker \zeta_1 \cap \ker \zeta_2).$$

Let $p^q(x\colon X)$ denote the local Betti numbers in a space X.

COROLLARY. *If* $(x\colon X;\ X_1, X_2)$ *is a proper local triad, and if the local Betti numbers are equal to the corresponding ranks of the homology groups in a point, then*

$$p^q(x\colon X_1 \cup X_2) + p^q(x\colon X_1 \cup X_2) = p^q(x\colon X_1) + p^q(x\colon X_2) + r(x\colon N^q) + r(x\colon N^{q-1}),$$

if these numbers are finite.

Proof. In this case, the Mayer-Vietoris sequence of Betti groups in a point may be used to obtain information about local Betti numbers. In particular, if

$$\xrightarrow{\alpha} A \xrightarrow{\beta} B \xrightarrow{\gamma} C \xrightarrow{\delta} D \rightarrow$$

is an exact sequence of finitely generated vector spaces, then it can be verified that $\dim B = \dim A - \dim(\ker \beta) + \dim C - \dim(\operatorname{im} \delta)$. On setting

$$\dim A = p^q(x\colon X_1 \cup X_2),$$

$$\dim B = p^q(x\colon X_1) + p^q(x\colon X_2),$$

$$\dim C = p^q(x\colon X_1 \cup X_2),$$

$$\dim(\ker \beta) = \dim(\operatorname{im} \alpha) = r(x\colon N^q),$$

$$\dim(\operatorname{im} \delta) = r(x\colon N^{q-1}),$$

the result is obtained from Theorem 2.

2. Generalized manifolds with boundaries have been studied by P. A. White [4], [5]. They are defined in the manner of generalized manifolds, in terms of local conditions, as follows.

A locally compact space M and a closed subset K is a *generalized manifold* M *with boundary* K (n-gm with boundary K), if

1) $M = K \cup A$, where A is open, $K = \overline{A} - A$, $\dim K < n$, and $\dim M = n$,
2) $p^r(x\colon M, K) = 0$ for all $x \in M$ and all $r \leq n - 1$,
3) $p^n(x\colon M, K) = 1$ for all $x \in M$,
4) $p^r(x\colon M) = 0$ for all $x \in K$ and all $r \leq n$.

The following theorem, a generalization of a theorem by White [3], is a consequence of the corollary in the first section.

THEOREM 3. *Let* M_1 *and* M_2 *be closed* n-gm's *with boundaries* K_1 *and* K_2, *respectively. Let* $M_1 \cap M_2 \subset K_1 \cap K_2$, *and let* $M_1 \cap M_2$ *be an* $(n - 1)$-gm. *Then* $M_1 \cup M_2$ *is an* n-gm *with boundary.*

Proof. Let $M_1 \cup M_2 = M$, $K = (K_1 \cup K_2) - \operatorname{Int}(M_1 \cap M_2)$. Let $A = M - K$; this is open since K is closed. Clearly, $\dim K < n$ and $\dim M = n$. The conditions on local Betti numbers are satisfied for all points not on $M_1 \cap M_2$.

To verify that $(x\colon M;\ M_1, M_2)$ is a proper local triad, consider the maps

$$k_2\colon (x\colon M_1, M_1 \cap M_2) \rightarrow (x\colon M_1 \cup M_2, M_2),$$

$$k_1: (x: M_2, M_1 \cap M_2) \rightarrow (x: M_1 \cup M_2, M_1).$$

That these induce isomorphisms onto in the Betti groups in a point follows from Wallace ([3], p. 182), where it is proved that the Map Excision Theorem holds for pairs such as those above if the spaces are locally compact Hausdorff and if the sets M_1, M_2 and $M_1 \cap M_2$ are closed and have compact boundary. Compact boundary may be assumed for computation of local homology groups in a point in locally compact spaces. Thus k_1 and k_2 are isomorphisms onto.

The remaining conditions are obtained by using the formula of the corollary in the following three cases.

1) $q = n$. Set $A = M_1 \cap M_2$; then by definition

$$p^q(x: A) = \text{rank} \lim_{\rightarrow} H^q(x: A \cup (X - P_\alpha), X - P_\alpha).$$

But by the Map Excision Theorem this is the same as $\text{rank} \lim_{\rightarrow} H^q(x: A, A - P_\alpha)$. Since $M_1 \cap M_2 = A$ is $(n - 1)$-dimensional, $p^n(x: M_1 \cap M_2) = 0$, and

$$p^n(x: M_1) = p^n(x: M_2) = 0$$

from 4) of the definition. Also $r(x: N^n) = 0$, since K is $(n - 1)$-dimensional. Finally $r(x: N^{n-1}) = 1$ since $p^{n-1}(x: K) = 1$, which is verified by noticing that $p^{n-1}(x: M_1) = p^{n-1}(x: M_2) = 0$, from 4) of the definition, and on examination of the effect of these values in the sequences of homology groups in a point of the pairs $(x: M_1, K)$ and $(x: M_2, K)$. Substitution of these values in the formula yields $p^n(x: M_1 \cup M_2) = 1$.

2) $q = n - 1$. Here $p^{n-1}(x: M_1 \cap M_2) = p^{n-1}(x: K) = 1$ and $r(x: N^{n-1}) = 1$, as above. All other numbers on the right are 0, and this implies that

$$p^{n-1}(x: M_1 \cup M_2) = 0.$$

3) $q < n - 1$. All terms can be seen to be 0.

From 1), 2), 3), it can be seen that each point on $M_1 \cap M_2$ has spherelike local Betti numbers, and that $M_1 \cup M_2$ is an n-gm with boundary K.

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