ON A THEOREM OF MAHLO CONCERNING ANTI-HOMOGENEOUS SETS

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Let us call a (linearly) ordered set *anti-homogeneous* if no two of its elements have the same character. J. Novák [7] has asked if there exists a continuous anti-homogeneous set. (I am indebted to M. Henriksen for calling Novák's question to my attention.) A partial answer had already been obtained by Mahlo [6], who had proved that, even under considerably weaker conditions, such a set must have a fantastically large cardinal—in fact, one whose existence is highly doubtful. In the present note, we describe Mahlo's result and derive some additional necessary conditions of our own.

An infinite ordered set is called *dense* if between any two of its elements there lies another [5, p. 90]. Let $F = (F_\pi)$ be a family of subsets of a dense set E, and define

$$F(\tau) = \bigcup_{\pi \leq \tau} F_\pi .$$

We assume that there exists an ordinal τ such that $F(\tau)$ is not nowhere dense in E; the least such τ will be denoted by $\phi = \phi(F)$.

LEMMA 1. *If* F_ϕ *is nowhere dense in* E, *then* ϕ *is a (nonzero) limit ordinal.*

Proof. Since F_ϕ is nowhere dense, while $F(\phi)$ is not, we have $\phi \neq 0$. Let I be an interval in which $F(\phi)$ is everywhere dense. If $\phi = \pi + 1$, there exists a subinterval J of I that does not meet $F(\pi)$; but then F_ϕ is everywhere dense in J.

We recall that $\mathrm{cf}(\alpha)$ denotes the smallest ordinal β such that ω_α is cofinal with ω_β; hence $\mathrm{cf}(\alpha) \leq \alpha$. If $\mathrm{cf}(\alpha) = \alpha$, then $\aleph_\alpha$ and ω_α are said to be *regular;* otherwise, they are *singular*. Every $\omega_{\mathrm{cf}(\alpha)}$ is regular. If ω_α is singular, then α is a limit ordinal. A *regular limit* cardinal $\mathfrak{p} = \aleph_\lambda > \aleph_0$ and its initial ordinal ω_λ are said to be *inaccessible*. Even the smallest such $\mathfrak{p}$ —if any exist—is of "exorbitant" magnitude [5, p. 131]. For example, not only are there $\mathfrak{p}$ cardinals less than $\mathfrak{p}$ (that is, $\omega_\lambda = \lambda$), but there exist $\mathfrak{p}$ cardinals $\mathfrak{q}$ less than $\mathfrak{p}$ such that there are $\mathfrak{q}$ cardinals less than $\mathfrak{q}$.

By a *segment* of a sequence $(\mu_\xi)_{\xi<\lambda}$ $(\lambda > 0)$ is meant any subsequence of the form $(\mu_\xi)_{\xi<\tau}$, where $0 < \tau < \lambda$. Inaccessible numbers $\aleph_\lambda$ and ω_λ are called *ρ-numbers* if

(1) every increasing sequence whose limit is ω_λ $(= \lambda)$ has a segment (of limit type) whose limit is inaccessible.

This concept was introduced by Mahlo [6]. The first inaccessible number pales into insignificance in comparison with the first ρ-number $\mathfrak{p}$. For example, not only are there $\mathfrak{p}$ inaccessible numbers less than $\mathfrak{p}$, but $\mathfrak{p}$ inaccessible numbers $\mathfrak{q}$ less than $\mathfrak{p}$ such that there are $\mathfrak{q}$ inaccessible numbers less than $\mathfrak{q}$. A sobering thought is that, conceivably, the cardinal $2^{\aleph_0}$ is a ρ-number. For additional discussion, see [2, Definition 3.7 ff.].

Received March 12, 1956.
Research supported in part by the National Science Foundation.

If α is a limit ordinal, then ω_α is cofinal with α, and $\omega_{cf(\alpha)} \leq \alpha$.

LEMMA 2. *A limit ordinal* λ *is a* ρ*-number if and only if it satisfies* (1).

Proof. The necessity follows directly from the definition of ρ-number. For the sufficiency, suppose that ω_λ is singular. Then $cf(\lambda) < \lambda$, so there exists an increasing sequence $(\mu_\xi)_{\xi<\omega_{cf(\lambda)}}$ whose limit is λ, and where $\mu_0 \geq cf(\lambda)$. Let τ be any limit ordinal $<\omega_{cf(\lambda)}$, and define $\pi = \lim_{\xi<\tau} \mu_\xi$. Then

$$\omega_{cf(\pi)} \leq \tau < \omega_{cf(\lambda)} \leq \omega_{\mu_0} < \omega_\pi,$$

so π cannot be inaccessible. Thus (1) fails. Therefore ω_λ must be regular, hence inaccessible.

Let x be an element or gap of an ordered set E. If the sets A [B] of all predecessors [successors] of x in E are nonempty and have no last [first] element, then A [the inverse of B] is cofinal with a unique regular initial ordinal, say ω_α [ω_β]; then x is said to have the *character* $c_{\alpha\beta}$. If $\alpha = \beta$, then x is *symmetric.*

The set E is *continuous* if it is dense and has no gaps [5, p. 90]. Every interval of a continuous set contains symmetric elements [5, p. 142]. The set of all $c_{\pi\pi}$-elements will be denoted by S_π.

The following result is a generalization of [6, Theorem 17]; because Mahlo's proof is hard to follow, we include a complete proof here.

THEOREM 1. *Let* E *be continuous. If* $F_\pi \supset S_\pi$ *for every* π, *and if* F_ϕ *is nowhere dense in* E, *then* ϕ *is a* ρ*-number.*

Proof. By Lemma 1, ϕ is a limit ordinal. Hence, if ϕ is not a ρ-number, then by Lemma 2 there exists an increasing sequence (μ_ξ) whose limit is ϕ, but no segment of which has an inaccessible limit. The same properties are evidently shared by every cofinal subsequence; we may accordingly assume that the sequence (μ_ξ) is of type $\omega_{cf(\phi)}$.

By definition of ϕ, $F(\phi)$ is everywhere dense in some interval I of E, while for every $\tau < \phi$, $F(\tau)$ is nowhere dense in E. Let J be an arbitrary subinterval of I. Let K_0 be a subinterval of J that is free of elements of $F(\mu_0)$. Let τ be any nonzero ordinal $<\omega_{cf(\phi)}$, and suppose that a sequence of intervals $(K_\xi)_{\xi<\tau}$ has been defined such that, for every $\xi < \tau$, K_ξ is disjoint from $F(\mu_\xi)$, and for every ξ with $\xi + 1 < \tau$, the endpoints of $K_{\xi+1}$ lie in the interior of K_ξ.

If τ is isolated, then there exists an interval K_τ that does not meet $F(\mu_\tau)$ and whose endpoints lie in the interior of $K_{\tau-1}$.

Suppose that τ is a limit ordinal. Define $\pi = \lim_{\xi<\tau} \mu_\xi$. Then $\omega_{cf(\tau)} \leq \pi$. Hence if $cf(\tau) = \pi$, then $\omega_\pi \leq \pi$, and π is inaccessible, contrary to assumption. Therefore $cf(\tau) < \pi$. Choose $\delta < \tau$ such that $cf(\tau) < \mu_\delta$.

If the interval $K' = \bigcap_{\xi<\tau} K_\xi$ reduces to a single element x, then x is symmetric—in fact, $x \in S_{cf(\tau)}$. Now $S_{cf(\tau)} \subset F(cf(\tau)) \subset F(\mu_\delta)$; hence K' meets $F(\mu_\delta)$. But this is impossible, since K' is contained in K_δ. Therefore K' must be a nondegenerate interval. We select a subinterval K_τ of K' that is disjoint from $F(\mu_\tau)$.

We thus define K_ξ for all $\xi < \omega_{cf(\phi)}$. Put

$$K = \bigcap_{\xi<\omega_{cf(\phi)}} K_\xi.$$

Then K is disjoint from F_τ —and hence from S_τ —for all $\tau < \phi$. Hence if K is a nondegenerate interval, then it must meet F_ϕ (since $F(\phi)$ is everywhere dense in I). The other possibility is that K reduces to a single element y. Then $y \in S_{cf(\phi)}$. By the remark above, we must then have $cf(\phi) \geq \phi$. Then $cf(\phi) = \phi$, and $y \in S_\phi \subset F_\phi$. Thus, in either case, K meets F_ϕ; so J meets F_ϕ. Since J was arbitrary, it follows that F_ϕ is everywhere dense in I. This contradiction completes the proof of the theorem.

COROLLARY 1. *Let* E *be continuous. If* $F_\pi \supset S_\pi$ *for every* π, *and if* F_π *is nowhere dense in* E *whenever* ω_π *is singular, then* ω_ϕ *is regular.*

Proof. If ϕ is a ρ-number, then ω_ϕ is regular; and if ϕ is not a ρ-number, then by Theorem 1, F_ϕ is not nowhere dense.

Let $S = (S_\pi)$; and let $C = (C_\pi)$, where C_π denotes the set of all $c_{\alpha\beta}$-elements of E such that $\alpha \leq \pi$, $\beta \leq \pi$, and either $\alpha = \pi$ or $\beta = \pi$. Then $S_\pi \subset C_\pi$. We define

$$\sigma = \phi(S), \quad \gamma = \phi(C)$$

(these numbers exist, since the set of *all* symmetric elements is everywhere dense in E). Evidently, $\sigma \geq \gamma$.

The following corollary includes [6, Theorems 14 and 17].

COROLLARY 2. *Let* E *be continuous. Then both* ω_σ *and* ω_γ *are regular; and if* S_σ *is nowhere dense in* E, *then* σ *is a* ρ-*number, while if* C_γ *is nowhere dense in* E, *then* γ *is a* ρ-*number.*

Proof. Put $F = S$, $F = C$, respectively. Then the second conclusion is immediate from Theorem 1; and the first follows from Corollary 1 upon observing that if ω_π is singular, then, by definition of character, both S_π and C_π are empty, hence nowhere dense in E.

We shall denote the cardinal of E by $\aleph_\varepsilon$. Obviously, $\varepsilon \geq \sigma$. Thus it follows from Corollary 2 that if S_σ is nowhere dense in E, then $|E|$ is at least as large as the first ρ-number. Observe that the hypothesis here is (apparently) many stages weaker than Novák's condition of anti-homogeneity.

By an η_ξ-*set* is meant an ordered set that is neither coinitial nor cofinal with any subset of power less than $\aleph_\xi$, and such that every element [gap] has character $c_{\alpha\beta}$ with $\alpha \geq \xi$ and [or] $\beta \geq \xi$ [5, p. 181].

THEOREM 2. *Let* I *be any interval of the continuous set* E, *and let* π *be any ordinal* $< \sigma$. *Then* I *contains an* $\eta_{\pi+1}$-*set, and* $|I| \geq 2^{\aleph_\pi}$. *If, furthermore,* $\pi < \gamma$, *then* I *contains an* $\eta_{\pi+1}$-*set that is a subinterval of* I, *and* $|I| \geq 2^{\aleph_{\pi+1}}$.

Proof. By definition of σ, $S(\pi)$ is nowhere dense in E; hence there exists a subinterval J of I that does not meet $S(\pi)$. For every $c_{\alpha\alpha}$-element of J, we have $\alpha \geq \pi + 1$. Therefore, by [4, Theorem XXII], J contains an $\eta_{\pi+1}$-set—and by [5, pp. 181-182] (or [8, Theorem V]), every $\eta_{\pi+1}$-set is of power at least $2^{\aleph_\pi}$. Similarly, if $\pi < \gamma$, there exists a subinterval J of I that does not meet $C(\pi)$, whence every element of J has a character $c_{\alpha\beta}$ with both $\alpha \geq \pi + 1$ and $\beta \geq \pi + 1$. Therefore any interior subinterval of J is an $\eta_{\pi+1}$-set—and by [4, Theorem XXI] (or [3, Remark 2]), every continuous η_ξ-set is of power at least $2^{\aleph_\xi}$.

We call E *power-homogeneous* if every interval of E has the same cardinality as E. There is no essential restriction in assuming E to be power-homogeneous, inasmuch as every interval contains a power-homogeneous subinterval (e.g., one of minimal cardinality).

THEOREM 3. *If* E *is continuous and power-homogeneous, then* $|C(\tau)| \leq \aleph_\tau < \aleph_\varepsilon$ *implies* $\tau < \gamma$.

Proof. Let I be any interval of E. Then $|I| = \aleph_\varepsilon$, so $|C(\tau)| < |I|$. Let $y \in I - C(\tau)$; then y has character $c_{\alpha\beta}$ with, say, $\beta \geq \tau + 1$. Let $(x_\xi)_{\xi<\omega_\beta}$ be an increasing sequence in I with y as limit. Since ω_β is regular (by definition of character), and $|C(\tau)| \leq \aleph_\tau < \aleph_\beta$, there exists an ordinal $\delta < \omega_\beta$ such that the subinterval $[x_\delta, y]$ of I is free of elements of $C(\tau)$. It follows that $C(\tau)$ is nowhere dense in E. Thus, $\tau < \gamma$.

Let $\mathfrak{p} = \aleph_\lambda$ be inaccessible; we shall call $\aleph_\lambda$ and ω_λ *semi-strongly* inaccessible if, for all $\mathfrak{q} < \mathfrak{p}$, we have $2^{\mathfrak{q}} \leq \mathfrak{p}$. For any regular $\mathfrak{p}$, this latter condition is equivalent to the following: $\mathfrak{p}^{\mathfrak{q}} = \mathfrak{p}$ for all $\mathfrak{q} < \mathfrak{p}$ [2, proof of Lemma 3.2]. Under the *hypothesis of inaccessible numbers* proposed in [1], every inaccessible $\mathfrak{p}$ would clearly be semi-strongly inaccessible. See also [3, Theorem 3 and Remark 1]. (For *strongly* inaccessible, replace $\leq$ above by $<$; see [1;2] for discussion and references.)

On combining various of the preceding results, we obtain

THEOREM 4. *Let* E *be continuous and power-homogeneous. If*

(i) $|C_\pi| \leq \aleph_\pi$ *for all* $\pi < \varepsilon$, *then*

(a) $\varepsilon = \sigma = \gamma$,

(b) *for all* $\pi < \varepsilon$, *every interval of* E *has a subinterval that is an* $\eta_{\pi+1}$*-set,*

(c) *every subset of* E *of power less than* $\aleph_\varepsilon$ *is nowhere dense in* E, *and*

(d) *every interval of* E *contains* $\aleph_\varepsilon$ *symmetric elements.*

If also

(ii) S_ε *is nowhere dense in* E, *then*

(e) ε *is a semi-strongly inaccessible* ρ*-number, and*

(f) *there exist* $\aleph_\varepsilon$ *inaccessible numbers* $\tau < \varepsilon$ *such that* $|S(\tau)| = \aleph_\tau$.

Proof. As already observed, we have $\bar{\varepsilon} \geq \sigma \geq \gamma$ in any case. If (i) holds, then for every $\tau < \varepsilon$, since $C(\tau) = \bigcup_{\pi \leq \tau} C_\pi$, we have

$$(2) \qquad |C(\tau)| \leq \sum_{\pi \leq \tau} |C_\pi| \leq |\tau + 1| \cdot \aleph_\tau = \aleph_\tau;$$

Theorem 3 now implies that $\gamma \geq \varepsilon$. This proves (a); and (b) then follows from Theorem 2. Since no set of power $\aleph_\pi$ can be dense in an $\eta_{\pi+1}$-set, (c) follows at once from (b). Since the set of all symmetric elements is everywhere dense in E, (c) implies (d).

If (ii) holds, then ε is a ρ-number, by Corollary 2; and the rest of (e) follows from Theorem 2.

Finally, since $S(\varepsilon)$ is everywhere dense in E, while S_ε is nowhere dense, the set $S' = S(\varepsilon) - S_\varepsilon = \bigcup_{\pi<\varepsilon} S_\pi$ must be everywhere dense in E. By (c), $|S'| = \aleph_\varepsilon$. Since $S_\pi \subset C_\pi$, we find from (i) that $|S_\pi| < \aleph_\varepsilon$ for all $\pi < \varepsilon$. Therefore, since $\aleph_\varepsilon$ is regular, there exist $\aleph_\varepsilon$ ordinals $\pi < \varepsilon$ for which S_π is nonempty; let these be $(\pi_\xi)_{\xi<\varepsilon}$, in increasing order. Let α be any ordinal $<\varepsilon$, and suppose that inaccessible ordinals τ_ξ $(\xi < \alpha)$ have been defined as in (f). Define $\delta = \sup_{\xi<\alpha} (\tau_\xi+1)$; then $\delta < \varepsilon$, since ε is regular. Since ε is a ρ-number, there exists a limit ordinal λ, with $\delta < \lambda < \varepsilon$, such that $\lim_{\delta\leq\xi<\lambda}\pi_\xi$ is inaccessible—hence equal to λ. Since $S(\lambda) \supset \bigcup_{\xi<\lambda} S_{\pi_\xi}$, we have

$$|S(\lambda)| \geq \sum_{\xi<\lambda} |S_{\pi_\xi}| \geq \sum_{\xi<\lambda} 1 = |\lambda| = \aleph_\lambda.$$

Since $S(\lambda) \subset C(\lambda)$, it then follows from (2) that $|S(\lambda)| = \aleph_\lambda$. We define $\tau_\alpha = \lambda$.

Remark. We are so far unable to prove that if ε is a semi-strongly inaccessible ρ-number, then there does exist a continuous set of power $\aleph_\varepsilon$ that is anti-homogeneous (or that satisfies the weaker conditions above).

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