# A THEOREM ON SIMPLE CARDINAL ALGEBRAS

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## 1. INTRODUCTION

A *cardinal algebra,* as defined by Tarski [1], is an algebraic system which is closed under an operation of countable addition and which satisfies certain axioms abstracted from the common properties of such diverse algebraic systems as the algebras of cardinal numbers, sets, relations, and so forth.

In the framework of a cardinal algebra one can define the concept of an ideal; and from this one can build up a fairly extensive representation theory for such algebras. As is to be expected, the major building blocks in such a theory are the so-called *simple cardinal algebras*, that is, the algebras having no nontrivial proper ideals. The most obvious examples of simple cardinal algebras are the algebra of nonnegative real numbers closed by the addition of $\infty$, and its three subalgebras: the algebra of nonnegative integers, similarly closed; the two-element algebra $\{0, \infty\}$; and the trivial algebra $\{0\}$. Interestingly enough, these are the only *known* examples; however, no proof that there exist no others has been constructed. This paper does not settle the question; but it gives a sufficient (and trivially necessary) condition for a simple cardinal algebra to be one of the four algebras mentioned above. The condition appears particularly natural in the context of the general representation theory.

Before stating and proving our theorem we shall, for convenience, quote a few definitions and results of Tarski [1] which are required in the sequel.

## 2. DEFINITIONS AND KNOWN RESULTS

A *cardinal algebra* is defined to be an algebraic system which consists of an underlying set A, a binary operation +, and an operation $\Sigma$ of countably infinite rank

$$\mathfrak{A} = \langle A, +, \Sigma \rangle,$$

and which satisfies seven axioms. The first five axioms merely imply that $\Sigma$ is a generalization of +, that there exists a zero element 0, and that the operations are unrestrictedly commutative and associative. The last two axioms are more restrictive. Axiom VI states that if the same element can be written as a sum in two different ways, then the summands have a common subdivision. Axiom VII asserts the existence of a certain type of greatest lower bound.

VI. *If* $\sum_{i<\infty} a_i = \sum_{j<\infty} b_j$, *then there exist elements* $c_{ij}$ *in* A *such that* $a_i = \sum_{j<\infty} c_{ij}$ *and* $b_j = \sum_{i<\infty} c_{ij}$, *for each* i, j.

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VII. *If* $a_i = a_{i+1} + b_i$ *for each* i, *then there exists an element* a *in* A *such that* $a_n = a + \sum_{i<\infty} b_{n+1}$, *for each* n.

A cardinal algebra can be made into a partially ordered set by defining

$$a \leq b \text{ if and only if } b = a + x \text{ for some } x \text{ in } A.$$

A subset $B \subset A$ is known as an *ideal* in $\mathfrak{A}$ if it is closed under $\Sigma$ and if $a \leq b \in B$ implies that $a \in B$. A cardinal algebra $\mathfrak{A}$ is called *simple* if it has no nontrivial proper ideals.

We shall require the following results due to Tarski. (The numbers in parentheses refer to the numbering system of [1]).

THEOREM 2.1 (2.21). *If* a *and* b *are elements of a cardinal algebra such that* $na \leq nb$ *for some finite integer* n, *then* $a \leq b$.

THEOREM 2.2 (2.33). *If* $\sum_{i \leq k} a_i \leq a$ *for each integer* k, *then* $\sum_{i<\infty} a_i \leq a$.

We will use the symbol $\infty a$ to denote the sum $\sum_{i<\infty} a_i$, where each $a_i = a$.

THEOREM 2.3 (9.34). *A cardinal algebra is simple if and only if, for any two nonzero elements* x *and* y, $\infty x = \infty y$.

THEOREM 2.4 (9.35). *Let* $\mathfrak{A}$ *be a simple cardinal algebra containing an indecomposable element* (that is, a nonzero element a such that $a = x + y$ implies that $x = 0$ or $y = 0$). *Then* $\mathfrak{A}$ *is isomorphic with the algebra of nonnegative integers closed by the addition of* $\infty$.

THEOREM 2.5 (14.7). *If* $\mathfrak{A}$ *is a simply-ordered simple cardinal algebra, then* $\mathfrak{A}$ *is isomorphic to a subalgebra of the nonnegative real numbers closed by the addition of* $\infty$.

## 3. THE NEW THEOREM

*Let*

$$\mathfrak{A} = \langle A, +, \Sigma \rangle$$

*be a simple cardinal algebra in which the inequality* $a < \infty b$ *implies that* $a \leq nb$ *for some finite integer* n. *Then* $\mathfrak{A}$ *is isomorphic to a subalgebra of the algebra of nonnegative real numbers closed by the addition of* $\infty$.

*Proof.* We make the following assertion, valid in every simple cardinal algebra:

If x and y are nonzero elements of A, then there exists a nonzero element z such that $z \leq x$ and $z \leq y$.

To prove this, note that, by (2.3), $x \leq \infty x = \infty y$ and hence, by Axiom VI, $x = \Sigma x_i$, where each $x_i \leq y$. Choose z to be any nonzero $x_i$, and the assertion follows.

We may obviously assume that A contains more than two elements, and furthermore that it contains no indecomposable element, since otherwise (2.4) completes the proof.

Let $a_0$ be an arbitrary nonzero element with $a_0 < \infty a_0$. Since $a_0$ is not indecomposable, there exist nonzero elements $x$ and $y$ such that $x + y = a_0$. By our assertion above, there exists a nonzero element $a_1$ with $a_1 \leq x$ and $a_1 \leq y$. Hence $2a_1 \leq a_0$. By repeating the argument, we obtain a monotone decreasing sequence of nonzero elements

$$a_0 > a_1 > \cdots > 0$$

such that $2a_{i+1} \leq a_i$, for each $i < \infty$.

We next assert that if $c$ is any nonzero element, then $a_n \leq c$ for some index $n$. Since $a_0 < \infty a_0 = \infty c$, we may choose $n$ so that $a_0 \leq nc$, by the hypothesis of the theorem. Then

$$na_n \leq 2^n a_n \leq 2^{n-1} a_{n-1} \leq \cdots \leq a_0 \leq nc .$$

Our assertion now follows from (2.1).

To show that $\mathfrak{A}$ is isomorphic to the nonnegative real numbers, it is sufficient to show that it is simply-ordered, in view of (2.5). Choose elements $a$ and $b$ in $A$ arbitrarily, and without loss of generality assume that $a < \infty a$ and $b < \infty b$. Let

$$K = \{ x \mid x \in A, x \leq a, x \leq b \} .$$

Choose the integer $n_0$ so that $n_0 a_0 \in K$, $(n_0 + 1)a_0 \notin K$. In general, for $k < \infty$, choose the integer $n_k$ so that

$$\sum_{i \leq k} n_i a_i \in K, \quad a_k + \sum_{i \leq k} n_i a_i \notin K .$$

Since $a < \infty a = \infty a_k$ and $b < \infty b = \infty a_k$, by the hypothesis of the theorem all these integers $n_k$ exist and are finite (some may possibly be zero). Now define

$$u = \sum_{i < \infty} n_i a_i .$$

Since $\sum_{i \leq k} n_i a_i \in K$, $\sum_{i \leq k} n_i a_i \leq a$ and $\sum_{i \leq k} n_i a_i \leq b$ for each $k < \infty$, whence $u \leq a$ and $u \leq b$ by (2.2). Consequently $u$ belongs to $K$.

Now suppose that $u < a$ and $u < b$; then by definition there exist nonzero elements $x$ and $y$ such that

$$a = u + x, \quad b = u + y .$$

By our two previous assertions, however, there exists an $a_k$ such that $a_k \leq x$ and $a_k \leq y$. Hence

$$u + a_k \leq u + x = a \text{ and } u + a_k \leq u + y = b .$$

Consequently

$$a_k + \sum_{i \leq k} n_i a_i \leq a_k + u \leq a$$

and

$$a_k + \sum_{i \leq k} n_i a_i \leq a_k + u \leq b,$$

which implies that $a_k + \sum_{i \leq k} n_i a_i \in K$, contradicting our choice of $n_k$. It follows that either $u = a \leq b$ or $u = b \leq a$. Thus $\mathfrak{A}$ is simply-ordered and, by (2.5), isomorphic to the nonnegative reals (plus $\infty$). This completes the proof of the theorem.

## REFERENCE

1. A. Tarski, *Cardinal algebras*, Oxford University Press, 1949.

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