ON COMPACT ABELIAN SEMI-GROUPS

by

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The purpose of the present paper is to extend a result on homogroups by Thierrin [4].

A semi-group is an algebraic system of elements $a, b, c, \ldots$ closed under a multiplicative binary operation satisfying $a(bc) = (ab)c$. Following Thierrin, we shall define homogroups, a class of semi-groups:

DEFINITION. A semi-group S is called a homogroup, if

(1) S contains an idempotent e,
(2) for each $a \in S$, there exists an element a' such that $aa' = e$,
(3) for every $a \in S$, $ea = ae$.

Such an idempotent e satisfying (1), (2) and (3) is unique. (See Thierrin). If a semi-group S has a zero element 0, then $a0 = 0a = 0$; the idempotent 0 satisfies the conditions (1), (2) and (3). Thus any semi-group having a zero is a homogroup.

DEFINITION. A topological semi-group is a semi-group with a Hausdorff topology in which multiplication is continuous in both variables.

Thierrin [4] has proved that every finite abelian semi-group is a homogroup. We shall establish the following result.

THEOREM. *Every compact abelian semi-group is a homogroup.*

Proof: If the semi-group S has a zero, the theorem is obvious from the remark above. Suppose that S has no zero element. Let us consider the family of all closed ideals[1] in S. The family is not empty, since S is a closed ideal. By Zorn's lemma and Cantor's intersection theorem, there exists at least one non-empty closed minimal ideal M. Let $a \in M$; then aM is an ideal and it is closed since S is compact. Clearly $aM \subset M$. This shows that $aM = M$ and $M \cdot M = M$ by the minimality assumption. Thus M is a group. Hence S contains the group M, which is an ideal of S. By a theorem of Thierrin [4], S is a homogroup.

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[1] For definition of ideal, see Clifford [1].

COROLLARY 1. *Every finite abelian semi-group is a homogroup.*

COROLLARY 2. *Every finite cyclic semi-group is a homogroup.* (Thierrin, [3]).

Any homogroup S with two-sided cancellation law is a group. For $e^2 = e$ implies $(ae)e = ae$ and $ae = a$ by the cancellation law. Similarly, $ea = a$. Hence S is a group. The following conclusion is immediate.

COROLLARY 3. *Every compact abelian semi-group with cancellation law is a compact group.*

This corollary has been proved for the non-abelian case. (For proof, see Gelbaum, Kalish and Olmsted [2] pp. 813-814, and especially Lemma 2.)

BIBLIOGRAPHY

1. A. H. Clifford, *Semi-groups containing minimal ideals*, Amer. J. Math. vol. 70 (1948) pp. 521-526.

2. B. Gelbaum, G. K. Kalish and J. M. H. Olmsted, *On the embedding of topological semi-groups and integral domains*. Proc. Amer. Math. Soc. vol. 2 (1951) pp. 807-821.

3. G. Thierrin, *Sur les homogroupes*, C. R. Acad. Sci. Paris vol. 234 (1952) pp. 1519-1521.

4. __________, *Sur quelques classes de demi-groupes*, C. R. Acad. Sci. Paris, vol. 236 (1953) pp. 33-35.

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A CONTRIBUTION TO THE THEORY OF MANIFOLDS

by

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1. INTRODUCTION. In a previous paper[1] [17], we introduced the local homology and homotopy invariants $\mathfrak{C}_v^r(x)$, $\mathfrak{C}_\sim^r(x)$ at the point x in a topological space M. We now use these invariants to define "manifolds" $\mathcal{M}^n(\mathfrak{C}_\sim)$, using homotopy; $\mathcal{M}^n(\mathfrak{C}_v, G)$, using Vietoris cycles with coefficients in G; and $\mathcal{M}^n(\mathfrak{C}_c, G)$, using Čech cycles over G. The three types are locally compact, metric, separable, n-dimensional topological spaces. We prove:

(A) Every $\mathcal{M}^n(\mathfrak{C}_\sim)$ is an $\mathcal{M}^n(\mathfrak{C}_v, I)$ where I is the group of integers; the converse is false.

(B) If Q is the field of rationals, then every $\mathcal{M}^n(\mathfrak{C}_v, I)$ is an $\mathcal{M}^n(\mathfrak{C}_v, Q)$, and its global integer and rational Betti numbers are identical in each dimension; the converse is false.

(C) If G is discrete, then every $\mathcal{M}^n(\mathfrak{C}_v, G)$ is an $\mathcal{M}^n(\mathfrak{C}_c, G)$, and conversely.

(D) If G is a field, then every $\mathcal{M}^n(\mathfrak{C}_c, G)$ is an n-gm, i.e., a "generalised manifold" in the sense of Wilder ([13], VIII, 1, p. 244); and conversely.

(E) If $\mathcal{M}^n(\mathfrak{C}_v, I)$ is compact and orientable, then its global integer Betti numbers satisfy a Poincaré Duality.

The proof of (A) requires Theorems 7.5, 7.6 of LTI, and a lemma of a technical nature concerning Vietoris cycles, proved in Section 4. That of (B) requires a lemma, proved in Section 5, concerning the integer and rational Betti numbers of certain compact sets. For (C), we shew in Section 2 how to convert concepts, expressed in terms of Vietoris cycles, into their Čech analogues; for this we use a theorem of Begle. In Section 3, we prove a rather complicated version of a "Theorem of Alexander Type", and use it to prove (D). By using known results proved

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[1]This reference will be denoted hereafter by LTI. Numbers not exceeding 13 refer to the Bibliography of LTI, and the rest to the Bibliography at the end of the present paper.