ON A THEOREM OF HENRY BLUMBERG

by

Casper Goffman

The theorem [1] asserts that for every real function $f(x)$ on $I = (0, 1)$ there is an everywhere dense set E such that $f(x)$ is continuous on E relative to E.

It seems plausible that an analogous result should hold for one-one transformations; i.e., that for every one-one correspondence $f(x)$, $f^{-1}(y)$ between $I = (0, 1)$ and $J = (0, 1)$ there should be sets E and $f(E)$, everywhere dense in I and J, respectively, such that $f(x)$, $f^{-1}(y)$ is a homeomorphism between them. The purpose of this note is to show that this is not true:

<u>There is a one-one correspondence</u> $f(x)$, $f^{-1}(y)$ <u>between</u> $I = (0, 1)$ <u>and</u> $J = (0, 1)$ <u>such that, for every</u> E <u>which is everywhere dense in</u> I, $f(x)$, $f^{-1}(y)$ <u>is not a homeomorphism between</u> E <u>and</u> $f(E)$.

Consider the following two sequences of subintervals of I. For every positive integer n and every $m = 0, 1, \ldots, 2^n - 1$, let $I_{nm} = (\frac{m}{2^n}, \frac{m+1}{2^n})$. Let $0 < a_1 < \cdots < a_k < \cdots < 1$ be an increasing sequence of positive numbers which converges to 1. Label the semi-open intervals $(0, a_1], (a_1, a_2], \cdots, (a_n, a_{n+1}], \cdots$ as $\tilde{I}_{11} = (0, a_1]$, $\tilde{I}_{12} = (a_1, a_2]$, $\tilde{I}_{21} = (a_2, a_3]$, $\cdots$, so that there is an $\tilde{I}_{nm}$ for every positive n and $m = 0, 1, \cdots, 2^n - 1$. The intervals $\tilde{I}_{nm}$ are mutually disjoint, their union is I, and if $n_1 > n_2$ or if $n_1 = n_2$, $m_1 > m_2$ then $\tilde{I}_{n_1 m_1}$ is to the right of $\tilde{I}_{n_2 m_2}$. Consider also the two sequences J_{nm} and $\tilde{J}_{nm}$ of subintervals of J obtained in the same way..

Now, for each positive n and $m = 0, 1, \cdots, 2^n - 1$, let $S_{nm} \subset I_{nm}$ and $T_{nm} \subset J_{nm}$ be non-empty, perfect, nowhere dense sets such that the sets S_{nm} are mutually disjoint and the sets T_{nm} are mutually disjoint. Let $S = \cup S_{nm}$ and $T = \cup T_{nm}$. Observe that both S and its complement intersect every subinterval of I in a set of cardinal number c, and that T has the same property relative to J. For each n and m, let $\tilde{S}_{nm} = \tilde{I}_{nm} - S$, $\tilde{T}_{nm} = \tilde{J}_{nm} - T$, and let $\tilde{S} = \cup \tilde{S}_{nm}$ and $\tilde{T} = \cup \tilde{T}_{nm}$.

It is clear that the $\tilde{S}_{nm}$, as well as the $\tilde{T}_{nm}$, are mutually disjoint, that $I = S \cup \tilde{S}$, $J = T \cup \tilde{T}$, and that every S_{nm}, $\tilde{S}_{nm}$, T_{nm} and $\tilde{T}_{nm}$ has cardinal number c. The correspondence $f(x)$, $f^{-1}(y)$ is defined by means of arbitrary one-one correspondences between S_{nm} and $\tilde{T}_{nm}$ and between $\tilde{S}_{nm}$ and T_{nm} for every n and m.

(a) Observe, first, that if $E \subset S$ is everywhere dense in a subinterval $I' \subset I$, then $f(x)$ is discontinuous at every point of $E \cap I'$ relative to E. For, if $x \in E \cap I'$, then since each S_{nm} is nowhere dense, there are $x_i \in E \cap I'$, $i = 1, 2, \ldots,$ such that $x_i \in S_{n_i m_i}$ and $x = \lim_{i\to\infty} x_i$, where $n_1 < n_2 < \cdots < n_i < \cdots$ and $0 \leq m_i \leq 2^{n_i} - 1$. But then $\lim_{i\to\infty} f(x_i) = 1 \neq f(x)$, since $f(x_i) \in J_{n_i m_i}$, for every i. By the same argument, if $E \subset T$ is everywhere dense in a subinterval $J' \subset J$, then $f^{-1}(y)$ is discontinuous at every point of $E \cap J'$ relative to E.

(b) Suppose next that $E \subset \widetilde{S}$ is everywhere dense in I. Then $E \cap \widetilde{I}_{nm}$ is non-empty so that $E \cap \widetilde{S}_{nm}$, and therefore $f(E) \cap T_{nm}$ is non-empty, for every n and m. Hence $f(E)$ meets every J_{nm} and so is everywhere dense in J.

The result of this note follows from (a) and (b). For, let E be everywhere dense in I and let $E_1 = E \cap S$ and $E_2 = E \cap \widetilde{S}$. If E_1 is not nowhere dense then, by (a), $f(x)$ has points of discontinuity on E_1 relative to E_1; hence on E relative to E. If E_1 is nowhere dense, then E_2 is everywhere dense in I so that, by (b), $f(E_2) \subset T$ is everywhere dense in J so that, by (a), $f^{-1}(y)$ is discontinuous on $f(E_2)$ relative to $f(E_2)$; hence on $f(E)$ relative to $f(E)$.

Finally, it is implicit in the proof that the given $f(x)$, if restricted to the set S, is not continuous on any everywhere dense subset $E \subset S$ relative to E. This shows that Blumberg's theorem really needs the second category properties of the space [2] on which $f(x)$ is defined. Since S has non-denumerable intersection with every interval, this shows:

<u>There is a set</u> S <u>which has non-denumerable intersection with every interval, and a real</u> $f(x)$ <u>defined on</u> S <u>such that for every</u> E <u>which is everywhere dense in</u> S, $f(x)$ <u>has a point of discontinuity on</u> E <u>relative to</u> E.

Bibliography

1. H. Blumberg, <u>New properties of all real functions</u>, Trans. Amer. Math. Soc., vol. 24 (1922), pp. 113-128.

2. H. D. Block and B. Cargal, <u>Arbitrary mappings</u>, Proc. Amer. Math. Soc., vol. 3, no. 6 (1952), pp. 937-941.

Wayne University
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