THE NATURAL MAP OF THE HYPERBOLIC PLANE INTO THE EUCLIDEAN CIRCLE

by

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The non-conformal Euclidean circle model of the hyperbolic plane, with straight lines represented by chords, was introduced into the literature by Beltrami in 1868 (1) (the well-known "psuedo-sphere" appeared four years later) and it made a contribution toward the general acceptance of the new geometry, which was still widely regarded as somehow "unreal". Beltrami discovered this circle model in the course of an investigation of the differential geometry of a surface of constant negative curvature, treated intrinsically. The projective model, with conic as "absolute," was introduced by Klein and Cayley in 1871 (2) - 1872 (3), and we now commonly think of the circle model as simply a special case of the projective. However, there exists a mapping of the hyperbolic plane into the Euclidean circle which is so simple and direct that it seems remarkable that it was not pointed out by one of the early non-Euclidean geometers. Had the modern concept of mathematical "model" been current at the time, it surely would have been noticed.

This mapping takes place in hyperbolic 3-space and makes use of the important theorem that the geometry on a horosphere ("limiting surface" — sphere of infinite radius) is Euclidean. It is defined as follows: In a hyperbolic 3-space of parameter p, let a plane m be tangent to a horosphere h at point O, and let each point of m map onto a point of h by projection along the corresponding axis of h.

It is elementary to observe that the entire plane m maps into the interior of a circle. Fig. 1 shows a section of the mapping in an axial plane of h which contains O. The arc MN is a segment of a horocycle on h, A' is a point of m, A'A is an axis of h, and A is the image of A' in the map. The axes through M and N are parallel to A'O and M and N are evidently limit points of the image points of the line A'O. On h, MN is a diameter of the circle g of image points. The points of any straight line on m are mapped by axes of h which determine an axial plane of h, and such planes cut h in horocycles, that is, in curves which in the Euclidean metric on h are straight lines. Every line of m is mapped into a chord of g.

Lobachevsky and Bolyai knew well the metric relations involved in this figure; in particular, the relation: $\tanh OA'/p = \overset{\frown}{OA}/p$. Since $\overset{\frown}{OM}$ is the image of an infinite segment of m, we have $1 = \overset{\frown}{OM}/p$; thus the radius of g is p.

Fig. 2 shows the Euclidean circle on h. We now make three geometrical observations, all evident by inspection of the mapping:

(1) Line pairs of m which are intersecting or parallel, are represented in the map by chords of g which meet respectively in the interior or on the circumference. Ultra-parallels are represented by chords which, if extended, either meet in the exterior or are Euclidean parallels.

(2) Angles at O are not distorted.

(3) On diameters of g, perpendicularity is not distorted--that is, if OAE (Fig. 2) is a right angle, then AE is the image of a line perpendicular to OA'.

To make the map a "model", we let the points of m be identified with their images. In the following paragraphs we shall use hyperbolic trigonometry to establish the polar construction for perpendiculars and to derive Euclidean expressions for the hyperbolic values of segments and angles. To keep the two metrics distinct, we shall write $\overline{AB}$ and $\overline{\Theta}$ for the Euclidean value and (AB) and (Θ) for the hyperbolic value of the indicated segment or angle. Thus, the relation mentioned above will be written:

$$\tanh\,(OA)/p = \overline{OA}/p.$$

In this model two lines are hyperbolic perpendiculars if they are conjugate in the polarity of g. To establish this fact we consider a quadrilateral in the position shown in Fig. 3, where P is the pole of KL. By Euclid we have $\overline{OL}\cdot\overline{OP} = p^2$ while $\cos\,(\Theta) = \cos\,(\overline{\Theta}) = \overline{OB}/\overline{OP} = \overline{OB}\cdot\overline{OL}/p^2$, or

$$\cos\,(\Theta) = \tanh\,(OB)/p\ \tanh\,(OL)/p.$$

But this is the trigonometric relation on the acute angle and adjacent sides of a tri-rectangle, and since the angles at L and B are right - angles in the hyperbolic metric, the angle at K must be also, which is what we wished to prove.

Returning now to Fig. 2, we note that AOE is the angle of parallelism of (OA) and the classical formulas $\tanh\,(OA)/p = \cos\,\pi\,(OA)$ can be checked by inspection. We also have:

$$\sinh\,(OA)/p = \cot\pi(OA) = \overline{OA}/\overline{AE}, \text{ and}$$
$$\cosh\,(OA)/p = \csc\pi(OA) = P/\overline{AE}$$

To find a Euclidean expression for the hyperbolic value of an arbitrary segment in the model, consider first the segment AB in Fig. 2. In the right triangle

AOB, we have, by hyperbolic trigonometry, $\tanh (AB)/p = \sinh (OA)/p \tan (\Theta)$. Using the relations above, this becomes:

$$\tanh (AB)/p = \overline{OA}/\overline{AE} \cdot \overline{AB}/\overline{OA} = \overline{AB}/\overline{AE}$$

By the use of the addition formulas, this expression gives a Euclidean ratio for the hyperbolic tangent of the hyperbolic value of any segment in the model. (It is a simple matter to show that this result agrees with the expression for distance given by Cayley-Klein in terms of cross-ratio).

To obtain a similar result for angles, we consider first the angle between a diameter and a chord such as angle φ in Fig. 4. By trigonometry we have $\cos (\varphi) = \sin (\Theta) \cosh \frac{(OB)}{p}$ which by the above relations becomes:

$$\cos (\varphi) = p \cos \overline{\varphi}/\overline{EB}.$$

Note that if P is on g, $\overline{EB}/p = \cos \overline{\varphi}$; the formulas then gives $\cos (\varphi) = 1$, and hence $(\varphi) = 0$ as we should expect. But suppose the lines are ultra-parallel so that the point P is exterior to g as in Fig. 3; in the trigonometry of the tri-rectnagle OBKL, we have $\cosh (KL)/p = \sin (\Theta) \cosh (OB)/p$ and precisely as before we have:

$$\cosh (KL)/p = p \cos \overline{\varphi}/\overline{EB}.$$

This result corresponds to the fact that in the analytic treatment of the hyperbolic plane the same invariant which gives the cosine of the angle made by intersecting lines, gives the hyperbolic cosine of the common normal of ultra-parallels.

With this interpretation for ultra-parallels, we have a Euclidean expression for the hyperbolic value of the angle made by any two secants of g; for the vertex can be joined to O and additional formulas employed if necessary.

Bibliography

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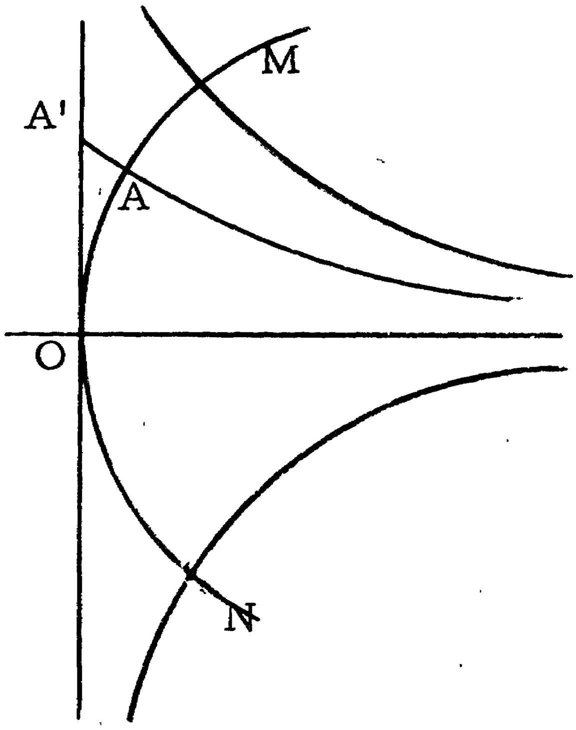

Figure 1

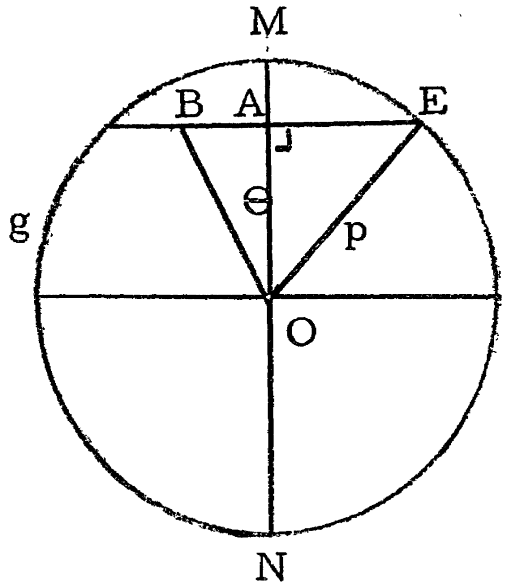

Figure 2

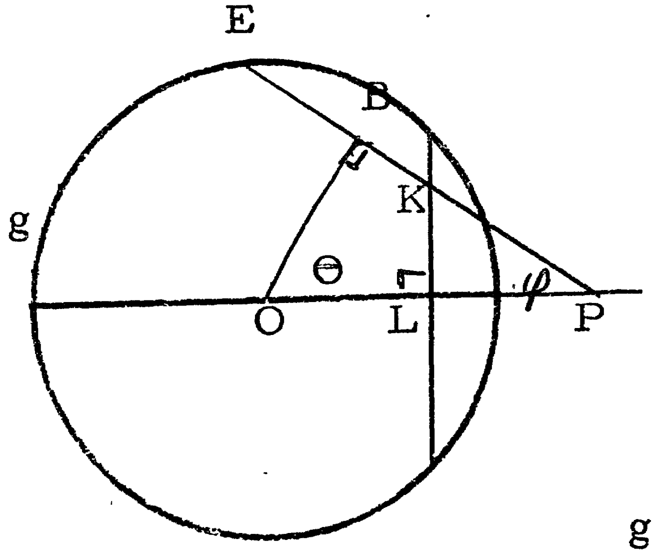

Figure 3

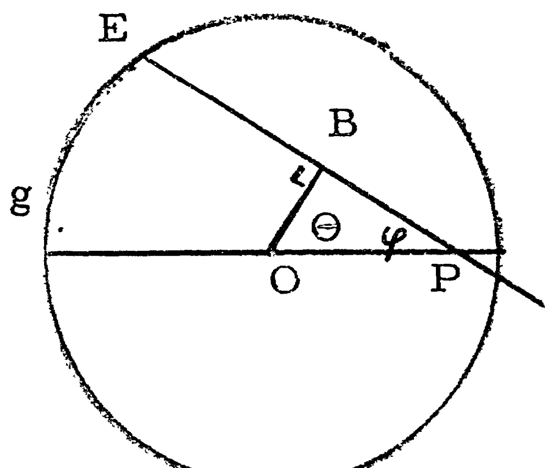

Figure 4